A new dynamical approach of Emden-Fowler equations and systems

Marie Françoise BIDAUT-VERON* Hector GIACOMINI†

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Abstract

We give a new approach on general systems of the form

$$(G)\begin{cases} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = -\operatorname{div}(|\nabla v|^{q-2}\nabla u) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases}$$

where $Q, p, q, \delta, \mu, s, m,\ a, b$ are real parameters, $Q, p, q \neq 1$, and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. In the radial case we reduce the problem to a quadratic system of four coupled first order autonomous equations, of Kolmogorov type. It allows to obtain new local and global existence or nonexistence results. We consider in particular the case $\varepsilon_1 = \varepsilon_2 = 1$. We describe the behaviour of the ground states in two cases where the system is variational. We give a result of existence of ground states for a nonvariational system with $p = q = 2$ and $s = m > 0$, that improves the former ones. It is obtained by introducing a new type of energy function. In the nonradial case we solve a conjecture of nonexistence of ground states for the system with $p = q = 2$, $\delta = m + 1$ and $\mu = s + 1$.

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*Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, 37200 Tours France. E-mail address:veronmf@univ-tours.fr

†Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, 37200 Tours France. E-mail address:Hector.Giacomini@lmpt.univ-tours.fr

1 Introduction

In this paper we consider the nonnegative solutions of Emden-Fowler equations or systems in $\mathbb{R}^N (N \geqq 1)$,

$$- \Delta_p u = - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^Q, \tag{1.1}$$

$$(G) \begin{cases} -\Delta_p u = - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = - \operatorname{div}(|\nabla v|^{q-2} \nabla u) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases} \tag{1.2}$$

where $Q, p, q, \delta, \mu, s, m, a, b$ are real parameters, $Q, p, q \neq 1$, and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. These problems are the subject of a very rich litterature, either in the case of source terms ($\varepsilon_1 = \varepsilon_2 = 1$) or absorption terms ($\varepsilon_1 = \varepsilon_2 = 1$) or mixed terms ($\varepsilon_1 = -\varepsilon_2$). In the sequel we are concerned by the radial solutions, except at Section 9 where the solutions may be nonradial.

In this article we we give a *new way of studying the radial solutions*. In Section 2 we reduce system (G) to a quadratic autonomous system:

$$(M) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z \left[N + a - sX - \delta Y - Z \right], \\ W_t = W \left[N + b - \mu X - mY - W \right], \end{cases}$$

where $t = \ln r$, and

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta \frac{u'}{|u'|^p}, \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m \frac{v'}{|v'|^q}. \tag{1.3}$$

This system is of Kolmogorov type. The reduction is valid for equations and systems with source terms , absorption terms , or mixed terms . It is remarkable that in the new system, p and q appear only as *simple coefficients*, which allows to treat any value of the parameters, even p or $q < 1$, and s, m, δ or $\mu < 0$.

In Section 3 we revisit the well-known scalar case (1.1), where (G) becomes two-dimensional. We show that the phase plane of the system gives *at the same time* the behaviour of the two equations

$$-\Delta_p u = |x|^a u^Q \text{ and } - \Delta_p u = - |x|^a u^Q,$$

which is a kind of *unification of the two problems*, with source terms or absorption terms. For the case of source term ($\varepsilon_1 = 1$), we find again the results of [2], [19], showing that the new dynamical approach is simple and does not need regularity results or energy functions. Moreover it gives a model for the study of system (G). Indeed if $p = q$, $a = b$ and $\delta + s = \mu + m$, system (G) admits solutions of the form (u, u), where u is a solution of (1.1) with $Q = \delta + s$.

In the sequel of the article we study the case of source terms, i.e. $(G) = (S)$, where

$$(S) \begin{cases} -\Delta_p u = |x|^a u^s v^\delta, \\ -\Delta_q v = |x|^b u^\mu v^m. \end{cases} \tag{1.4}$$

This system has been studied by many authors, in particular the Hamiltonian problem $s = m = 0$, in the linear case $p = q = 2$, see for example [20], [31], [29], [9], [33], [14], and the potential system where $\delta = m+1$, $\mu = s+1$ and $a = b$, see [7], [34], [35]; the problem with general powers has been studied in [3], [39], [40], [41] in the linear case and [6], [12], [42] in the quasilinear case, see also [1], [10], [13].

Here we suppose that $\delta, \mu > 0$, so that the system is always coupled, $s, m \geqq 0$, and we assume for simplicity

$$1 < p, q < N, \qquad \min(p+a, q+b) > 0, \qquad D = \delta\mu - (p-1-s)(q-1-m) > 0. \tag{1.5}$$

We say that a positive solution (u, v) in $(0, R)$ is <u>regular</u> at 0 if $u, v \in C^2(0, R) \cap C([0, R))$. Condition $\min(p+a, q+b) > 0$ guaranties the existence of local regular solutions. Then $u, v \in C^1([0, R))$. when $a, b > -1$, and $u'(0) = v'(0) = 0$. The assumption $D > 0$ is a classical condition of superlinearity for the system.

We are interessed in the existence or nonexistence of <u>*ground states*</u>, called G.S., that means global positive (u, v) in $(0, \infty)$ and regular at 0. We exclude the case of "trivial" solutions, $(u, v) = (0, C)$ or $(C, 0)$, where C is a constant, which can exist when $s > 0$ or $m > 0$.

In Section 4 we give a series of <u>*local existence or nonexistence*</u> results concerning system (S), which complete the nonexistence results found in the litterature. They are not based on the fixed point method, quite hard in general, see for example [19], [27]. We make a dynamical analysis of the linearization of system (M) near each fixed point, which appears to be performant, even for the regular solutions. For a better exposition, the proofs are given at Section 10.

In Section 5 we study the <u>*global existence*</u> of G.S. This problem has been often compared with the nonexistence of positive solutions of the Dirichlet problem in a ball, see [29], [30], [12], [13]. Here we use a shooting method adapted to system (M), which allows to avoid questions of regularity of system (S). We give a new way of comparison, and improve the former results:

Theorem 1.1 *(i) Assume $s < \frac{N(p-1)+p+pa}{N-p}$ and $m < \frac{N(q-1)+q+qb}{N-q}$. If system (S) has no G.S., then*

(i) there exist regular radial solutions such that $X(T) = \frac{N-p}{p-1}$ and $Y(T) = \frac{N-q}{q-1}$ for some $T > 0$, with $0 < X < \frac{N-p}{p-1}$ and $0 < Y < \frac{N-q}{q-1}$ on $(-\infty, T)$.

(ii) there exists a positive radial solution (u, v) of the Dirichlet problem in a ball $B(0, R)$.

This result is a key tool in the next Sections for proving the existence of a G.S. It gives also new existence results for the Dirichlet problem, see Corollary 5.3. We also give a complementary result:

Proposition 1.2 *Assume $s \geqq \frac{N(p-1)+p+pa}{N-p}$ and $m \geqq \frac{N(q-1)+q+qb}{N-q}$. Then all the regular radial solutions are G.S.*

In Section 6 we study the <u>*radial*</u> solutions of the well known Hamiltonian system

$$(SH) \left\{ \begin{array}{l} -\Delta u = |x|^a v^\delta, \\ -\Delta v = |x|^b u^\mu, \end{array} \right.$$

corresponding to $p = q = 2 < N$, $s = m = 0$, $a > -2$, which is variational. In the case $a = b = 0$, a main conjecture was made in [32]:

Conjecture 1.3 *System* (SH) *with* $a = b = 0$ *admits no (radial or nonradial) G.S. if and only if* (δ, μ) *is under the hyperbola of equation*

$$\frac{N}{\delta+1} + \frac{N}{\mu+1} = N - 2.$$

The question is still open; it was solved in the radial case in [26], [29], then partially in [31], [9], and up to the dimension $N = 4$ in [33], see references therein. Here we find again and extend to the case $a, b \neq 0$ some results of [20] relative to the G.S., with a shorter proof. We also give an existence result for the Dirichlet problem improving a result of [14].

Theorem 1.4 *Let* $\mathcal{H}_0$ *be the critical hyperbola in the plane* (δ, μ) *defined by*

$$\frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} = N - 2. \tag{1.6}$$

Then

(i) System (SH) *admits a (unique) radial G.S. if and only if* (δ, μ) *is above* $\mathcal{H}_0$ *or on* $\mathcal{H}_0$*.*

(ii) The radial Dirichlet problem in a ball has a solution if and only if (δ, μ) *is under* $\mathcal{H}_0$*.*

(iii) On $\mathcal{H}_0$ *the G.S. has the following behaviour at* ∞ *: assuming for example* $\delta > \frac{N+a}{N-2}$*, then* $\lim_{r\to\infty} r^{N-2}u(r) = \alpha > 0$*, and*

$$\begin{aligned}
\lim_{r\to\infty} r^{(N-2)\mu-(2+b)}v = \beta > 0 \qquad & \text{if } \mu < \frac{N+b}{N-2}, \\
\lim_{r\to\infty} r^{N-2}v = \beta > 0 \qquad & \text{if } \mu > \frac{N+b}{N-2}, \\
\lim_{r\to\infty} r^{N-2}\left|\ln r\right|^{-1} v = \beta > 0 \qquad & \text{if } \mu = \frac{N+b}{N-2}.
\end{aligned}$$

Our proofs use a Pohozaev type function; in terms of the new variables X, Y, Z, W, it contains a quadratic factor

$$\begin{aligned}
\mathcal{E}_H(r) &= r^N \left[u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1}\frac{vu'}{r} + \frac{N+b}{\mu+1}\frac{uv'}{r} \right] \\
&= r^{N-2}uv \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right].
\end{aligned} \tag{1.7}$$

As observed in ([20]) the G.S. can present a non-symmetric behaviour. This non-symmetry phenomena has to be taken in account for solving conjecture (1.3).

In Section 7 we consider the radial solutions of a <u>nonvariational system</u>:

$$(SN) \left\{ \begin{array}{l} -\Delta u = |x|^a u^s v^\delta, \\ -\Delta v = |x|^a u^\mu v^s, \end{array} \right.$$

where $p = q = 2 < N$, $a = b > -2$ and $m = s > 0$. For small s it appears as a perturbation of system (SH). In the litterature very few results are known for such nonvariational systems. Our main result in this Section is a new result of <u>*existence of G.S.*</u> valid for any s:

Theorem 1.5 *Consider the system* (SN), *with* $N > 2$, $a > -2$. *We define a curve* $\mathcal{C}_s$ *in the plane* (δ, μ) *by*

$$\frac{N+a}{\mu+1} + \frac{N+a}{\delta+1} = N - 2 + \frac{(N-2)s}{2} \min(\frac{1}{\mu+1}, \frac{1}{\delta+1}), \tag{1.8}$$

located under the hyperbola defined by (1.6). If (δ, μ) *is above* $\mathcal{C}_s$, *system* (SN) *admits a G.S.*

This result is obtained by constructing a new type of energy function which contains two terms in X^2, Y^2 :

$$\begin{aligned}\Phi(r) &= r^N \left[u'v' + r^b \frac{u^{\mu+1}v^s}{\mu+1} + r^a \frac{u^s v^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1}\frac{vu'}{r} + \frac{N+b}{\mu+1}\frac{uv'}{r} + \frac{s}{2(\delta+1)}\frac{vu'^2}{u} + \frac{s}{2(\mu+1)}\frac{uv'^2}{v} \right] \\ &= r^{N-2}uv \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} + \frac{s}{2(\delta+1)}X^2 + \frac{s}{2(\mu+1)}Y^2 \right]. \end{aligned} \tag{1.9}$$

In Section 8 we consider the <u>*radial*</u> solutions of the potential system

$$(SP) \begin{cases} -\Delta_p u = |x|^a \, u^s v^{m+1}, \\ -\Delta_q v = |x|^a \, u^{s+1} v^m, \end{cases}$$

where $\delta = m+1, \mu = s+1$ and $a = b$, which is variational, see [34], [35]. Using system (M) we deduce new results of existence:

Theorem 1.6 *Let* $\mathcal{D}$ *be the critical line in the plane* (m, s) *defined by*

$$N + a = (m+1)\frac{N-q}{q} + (s+1)\frac{N-p}{p}.$$

Then

(i) System (SP) *admits a radial G.S. if and only if* (m, s) *is above or on* $\mathcal{D}$.

(ii) On $\mathcal{D}$ *the G.S. has the following behaviour: suppose for example* $q \leqq p$. *Let* $\lambda^* = N + a - (s+1)\frac{N-p}{p-1} - m\frac{N-q}{q-1}$. *Then* $\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u(r) = \alpha > 0$, *and*

$$\lim_{r\to\infty} r^{\frac{N-q}{q-1}} v(r) = \beta > 0 \qquad \text{if } \lambda^* < 0, \tag{1.10}$$

$$\lim_{r\to\infty} r^{\frac{\frac{N-p}{p-1}\mu-(q+b)}{q-1-m}} v(r) = \beta > 0 \qquad \text{if } \lambda^* > 0, \tag{1.11}$$

$$\lim_{r\to\infty} r^{\frac{N-q}{q-1}} |\ln r|^{-\frac{1}{q-1-m}} v(r) = \beta > 0 \qquad \text{if } \lambda^* = 0. \tag{1.12}$$

In particular (1.10) holds if $p = q$, *or* $q \leqq m+1$.

(iii) The radial Dirichlet problem in a ball has a solution if and only if (m, s) *is under* $\mathcal{D}$.

In that case we use the following energy function, which deserves to be compared with the one of Section 6 , since it has also a quadratic factor:

$$\mathcal{E}_P(r) = r^N\left[(s+1)(\frac{|u'|^p}{p'} + \frac{N-p}{p}\frac{u\,|u'|^{p-2}\,u'}{r}) + (m+1)(\frac{|v'|^q}{q'} + \frac{N-q}{q}\frac{v\,|v'|^{q-2}\,v'}{r}) + r^a u^{s+1}v^{m+1}\right]$$

$$= r^{N-2-a}\frac{|u'|^{p-1}\,|v'|^{q-1}}{u^s v^m}\left[ZW - \frac{(s+1)W(N-p-(p-1)X)}{p} - \frac{(m+1)Z(N-q-(q-1)Y)}{q}\right]. \tag{1.13}$$

Finally in Section 9 we deduce a <u>*nonradial*</u> result for the potential system in the case of two Laplacians:

$$(SL)\begin{cases} -\Delta u = |x|^a\, u^s v^{m+1}, \\ -\Delta v = |x|^a\, u^{s+1} v^m. \end{cases}$$

Our result proves a conjecture proposed in [7], showing that in the subcritical case there exists no G.S.:

Theorem 1.7 *Assume $a > -2$ and $s, m \geqq 0$. If*

$$s+m+1 < \min(\frac{N+2}{N-2}, \frac{N+2+2a}{N-2}), \tag{1.14}$$

then system (SL) admits no (radial or nonradial) G.S.

Our proof uses the estimates of [7], which up to now are the only extensions of the results of [18] to systems. It is based on the construction of a nonradial Pohozaev function extending the radial one given at (1.13) for $p = q = 2$, different from the energy function used in [7].

The case of the system (G) with absorption terms ($\varepsilon_1 = \varepsilon_2 = -1$) or mixed terms ($\varepsilon_1 = -\varepsilon_2 = 1$), studied in [4], [5], will be the subject of a second article. Our approach also extends to a system with gradient terms,

$$\begin{cases} -\Delta_p u = \varepsilon_1\,|x|^a\, u^s v^\delta\,|\nabla u|^\eta\,|\nabla v|^\ell\,, \\ -\Delta_q v = \varepsilon_2\,|x|^b\, u^\mu v^m\,|\nabla u|^\nu\,|\nabla v|^\kappa\,, \end{cases} \tag{1.15}$$

which will be studied in another work.

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2 Reduction to a quadratic system

2.1 The change of unknowns

Here we consider the radial positive solutions $r \mapsto (u(r), v(r))$ of system (G) on any interval (R_1, R_2), that means

$$\begin{cases} \left(|u'|^{p-2}\,u'\right)' + \frac{N-1}{r}\,|u'|^{p-2}\,u' = r^{1-N}\left(r^{N-1}\,|u'|^{p-2}\,u'\right)' = -\varepsilon_1 r^a u^s v^\delta, \\ \left(|v'|^{q-2}\,v'\right)' + \frac{N-1}{r}\,|v'|^{q-2}\,v' = r^{1-N}\left(r^{N-1}\,|v'|^{p-2}\,v'\right)' = -\varepsilon_2 r^b u^\mu v^m. \end{cases}$$

Near any point r where $u(r) \neq 0, u'(r) \neq 0$ and $v(r) \neq 0$, $v'(r) \neq 0$ we define

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta \left|u'\right|^{-p} u', \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m \left|v'\right|^{-q} v', \tag{2.1}$$

where $t = \ln r$. Then we find the system

$$(M) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z \left[N + a - sX - \delta Y - Z \right], \\ W_t = W \left[N + b - \mu X - mY - W \right]. \end{cases}$$

This sytem is *quadratic*, and moreover a very simple one, of *Kolmogorov type*: it admits four invariant hyperplanes: $X = 0, Y = 0, Z = 0, W = 0$. As a first consequence all the fixed points of the system are explicite. The trajectories located on these hyperplanes do not correspond to a solution of system (G); they will be called *nonadmissible*.

We suppose that the discriminant of the system

$$D = \delta\mu - (p-1-s)(q-1-m) \neq 0. \tag{2.2}$$

Then one can express u, v in terms of the new variables:

$$u{=}r^{-\gamma}(\left|X\right|^{p-1}\left|Z\right|)^{(q-1-m)/D}(\left|Y\right|^{q-1}\left|W\right|)^{\delta/D}, \qquad v{=}r^{-\xi}(\left|X\right|^{p-1}\left|Z\right|)^{\mu/D}(\left|Y\right|^{q-1}\left|W\right|)^{(p-1-s)/D}, \tag{2.3}$$

where γ and ξ are defined by

$$\gamma = \frac{(p+a)(q-1-m) + (q+b)\delta}{D}, \qquad \xi = \frac{(q+b)(p-1-s) + (p+a)\mu}{D}, \tag{2.4}$$

or equivalently by

$$(p-1-s)\gamma + p + a = \delta\xi, \qquad (q-1-m)\xi + q + b = \mu\gamma. \tag{2.5}$$

Since system (M) is autonomous, each admissible trajectory $\mathcal{T}$ in the phase space corresponds to a solution (u, v) of system (G) unique up to a scaling: if (u, v) is a solution, then for any $\theta > 0$, $r \mapsto (\theta^\gamma u(\theta r), \theta^\xi v(\theta r))$ is also a solution.

2.2 Fixed points of system (M)

System (M) has at most 16 fixed points. The main fixed point is

$$M_0 = (X_0, Y_0, Z_0, W_0) = (\gamma, \xi, N - p - (p-1)\gamma, N - q - (q-1)\xi), \tag{2.6}$$

corresponding to the particular solutions

$$u_0(r) = Ar^{-\gamma}, v_0(r) = Br^{-\xi}, \quad A, B > 0, \tag{2.7}$$

when they exist, depending on $\varepsilon_1, \varepsilon_2$. The values of A and B are given by

$$\begin{aligned}
A^D &= \left(\varepsilon_1\gamma^{p-1}(N-p-\gamma(p-1))\right)^{q-1-m}\left(\varepsilon_2\xi^{q-1}(N-q-(q-1)\xi)\right)^{\delta},\\
B^D &= \left(\varepsilon_2\xi^{q-1}(N-q-(q-1)\xi)\right)^{p-1-s}\left(\varepsilon_1\gamma^{p-1}(N-p-(p-1)\gamma)\right)^{\mu}.
\end{aligned}$$

The other fixed points are

$$\begin{aligned}
&0=(0,0,0,0),\quad N_0=(0,0,N+a,N+b),\quad A_0=(\frac{N-p}{p-1},\frac{N-q}{q-1},0,0),\\
&I_0=(\frac{N-p}{p-1},0,0,0),\quad J_0=(0,\frac{N-q}{q-1},0,0),\quad K_0=(0,0,N+a,0),\quad L_0=(0,0,0,N+b),\\
&G_0=(\frac{N-p}{p-1},0,0,N+b-\frac{N-p}{p-1}\mu),\quad H_0=(0,\frac{N-q}{q-1},N+a-\frac{N-q}{q-1}\delta,0),
\end{aligned}$$

and if $m \neq q-1$,

$$\begin{aligned}
P_0 &= (\frac{N-p}{p-1},\frac{\frac{N-p}{p-1}\mu-(q+b)}{q-1-m},0,\frac{(q-1)(N+b-\frac{N-p}{p-1}\mu)-m(N-q)}{q-1-m}),\\
C_0 &= \left(0,-\frac{q+b}{q-1-m},0,\frac{(N+b)(q-1)-m(N-q)}{q-1-m}\right),\\
R_0 &= \left(0,-\frac{q+b}{q-1-m},N+a+\delta\frac{b+q}{q-1-m},\frac{(N+b)(q-1)-m(N-q)}{q-1-m}\right),
\end{aligned}$$

and by symmetry, if $s \neq p-1$,

$$\begin{aligned}
Q_0 &= (\frac{\frac{N-q}{q-1}\delta-(p+a)}{p-1-s},\frac{N-q}{q-1},\frac{(p-1)(N+a-\frac{N-q}{q-1}\delta)-s(N-p)}{p-1-s},0),\\
D_0 &= \left(-\frac{p+a}{p-1-s},0,\frac{(N+a)(p-1)-s(N-p)}{p-1-s},0\right),\\
S_0 &= \left(-\frac{p+a}{p-1-s},0,\frac{(N+a)(p-1)-s(N-p)}{p-1-s},N+b+\mu\frac{a+p}{p-1-s}\right).
\end{aligned}$$

2.3 First comments

Remark 2.1 *This formulation allows to treat more general systems with signed solutions by reducing the study on intervals where u and v are nonzero. Consider for example the problem*

$$-\Delta_p u = \varepsilon_1 |x|^a |u|^s |v|^{\delta-1} v, \quad -\Delta_q v = \varepsilon_2 |x|^b |v|^m |u|^{\mu-1} u.$$

On any interval where $uv > 0$, the couple $(|u|, |v|)$ is a solution of (G). On any interval where $u > 0 > v$, the couple $(u, |v|)$ satisfies (G) with $(\varepsilon_1, \varepsilon_2)$ replaced by $(-\varepsilon_1, -\varepsilon_2)$.

Remark 2.2 *There is another way for reducing the system to an autonomous form: setting*

$$\mathrm{U}(t)=r^{\gamma}u,\quad \mathrm{V}(t)=r^{\xi}v,\quad \mathrm{H}(t)=-r^{(\gamma+1)(p-1)}\left|u'\right|^{p-2}u',\quad \mathrm{K}(t)=-r^{(\xi+1)(q-1)}\left|v'\right|^{q-2}v',$$

with $t = \ln r$, *we find*

$$\left\{ \begin{array}{ll} \mathrm{U}_t = \gamma\mathrm{U} - |\mathrm{H}|^{(2-p)/(p-1)}\,\mathrm{H}, & \mathrm{V}_t = \zeta\mathrm{U} - |\mathrm{K}|^{(2-q)/(q-1)}\,\mathrm{K}, \\ \mathrm{H}_t = (\gamma(p-1) + p - N)\mathrm{H} + \varepsilon_1\mathrm{U}^s\mathrm{V}^\delta, & \mathrm{K}_t = (\zeta(q-1) + q - N)\mathrm{K} + \varepsilon_2\mathrm{U}^\mu\mathrm{V}^m. \end{array} \right. \tag{2.8}$$

It extends the well-known transformation of Emden-Fowler in the scalar case when $p = 2$, *used also in [2] for general* p, *see Section 3. When* $p = q = 2$ *we obtain*

$$\left\{ \begin{array}{l} \mathrm{U}_{tt} + (N - 2 - 2\gamma)\mathrm{U}_t - \gamma(N - 2 - \gamma)\mathrm{U} + \varepsilon_1\mathrm{U}^s\mathrm{V}^\delta = 0, \\ \mathrm{V}_{tt} + (N - 2 - 2\xi)\mathrm{V}_t - \xi(N - 2 - \xi)\mathrm{V} + \varepsilon_2\mathrm{U}^\mu\mathrm{V}^m = 0, \end{array} \right. \tag{2.9}$$

which was extended to the nonradial case and used for Hamiltonian systems ($s = m = 0$), *with source terms in [9]* ($\varepsilon_1 = \varepsilon_2 = 1$) *and absorption terms in [4]* ($\varepsilon_1 = \varepsilon_2 = -1$). *Our system is more adequated for finding the possible behaviours: unlike system (2.8)it has no singularity, since it is polynomial, also its fixed points at* ∞ *are not concerned when we deal with solutions* $u, v > 0$.

Remark 2.3 *In the specific case* $p = q = 2$, *setting*

$$z = XZ = \varepsilon_1 r^{2+a} |u|^{s-2} u\, |v|^{\delta-1} v, \qquad w = YW = \varepsilon_2 r^{2+b} |u|^{\mu-1} u\, |v|^{m-2} v,$$

we get the following system

$$\left\{ \begin{array}{ll} X_t = X^2 - (N-2)X + z, & Y_t = Y^2 - (N-2)Y + w, \\ z_t = z\left[2 + a + (1-s)X - \delta Y\right], & w_t = w\left[2 + b - \mu X + (1-m)Y\right]. \end{array} \right.$$

It has been used in [20] for studying the Hamiltonian system (SH). *Even in that case we will show at Section 6 that system* (M) *is more performant, because it is of Kolmogorov type.*

Remark 2.4 *Assume* $p = q$ *and* $a = b$. *Setting* $t = k\hat{t}$ *and* $\left(\hat{X}, \hat{Y}, \hat{Z}, \hat{W}\right) = k(X, Y, Z, W)$, *we obtain a system of the same type with* N, a *replaced by* $\hat{N}, \hat{a}$, *with*

$$\frac{\hat{N} - p}{N - p} = k = \frac{\hat{N} + \hat{a}}{N + a}.$$

It corresponds to the change of unknowns

$$r = \hat{r}^k, \qquad \hat{u}(\hat{r}) = C_1 u(r), \quad \hat{v}(\hat{r}) = C_2 u(r), \qquad C_1 = k^{p(p-1-m+\delta)/D}, \quad C_2 = k^{(p(p-1-s+\mu))/D}.$$

From (2.3) and (2.4), we get $\hat{\gamma}/\gamma = \hat{\xi}/\xi = k = \frac{p+\hat{a}}{p+a}$. *There is one free parameter. In particular*

1) we get a system without power ($\hat{a} = 0$), *by taking*

$$\hat{N} = \frac{p(N+a)}{p+a}, \qquad k = \frac{p}{p+a};$$

2) we get a system in dimension $\hat{N} = 1$, *by taking*

$$k = -\frac{p-1}{N-p} < 0,, \qquad \hat{a} = \frac{p + a - (N+a)p}{N-p}.$$

3 The scalar case

We first study the signed solutions of two scalar equations with source or absorption:

$$-\Delta_p u = -r^{1-N}\left(r^{N-1}\left|u'\right|^{p-2}u'\right)' = \varepsilon\left|x\right|^a\left|u\right|^{Q-1}u, \tag{3.1}$$

with $\varepsilon = \pm 1$, $1 < p < N$, $Q \neq p-1$ and $p+a > 0$.

We cannot quote all the huge litterature concerning its solutions, supersolutions or subsolutions, from the first studies of Emden and Fowler for $p = 2$, recalled in [16]; see for example [2] and [37], for any $p > 1$, and references therein. We set

$$Q_1 = \frac{(N+a)(p-1)}{N-p}, \qquad Q_2 = \frac{N(p-1)+p+pa}{N-p}, \qquad \gamma = \frac{p+a}{Q+1-p}.$$

From Remark 2.4 we could reduce the system to the case $a = 0$, in dimension $\hat{N} = p(N+a)/(p+a)$. However we do not make the reduction, because we are motivated by the study of system (G), and also by the nonradial case.

3.1 A common phase plane for the two equations

Near any point r where $u(r) \neq 0$ (positive or negative), and $u'(r) \neq 0$ setting

$$X(t) = -\frac{ru'}{u}, \qquad Z(t) = -\varepsilon r^{1+a}\left|u\right|^{Q-1}u\left|u'\right|^{-p}u', \tag{3.2}$$

with $t = \ln r$, we get a 2-dimensional system

$$(M_{scal})\begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Z_t = Z\left[N+a-QX-Z\right]. \end{cases}$$

and then $|u| = r^{-\gamma}(\,|Z|\,|X|^{p-1})^{1/(Q+1-p)}$. This change of unknown was mentioned in [11] in the case $p = 2, \varepsilon = 1$ and $N = 3$. It is remarkable that system (M_{scal}) is the same for the *two cases* $\varepsilon = \pm 1$, the only difference is that $X(t)Z(t)$ has the sign of ε :

The equation with source ($\varepsilon = 1$) is associated to the 1^{st} and 3^{rd} quadrant. It is well known that any local solution has a unique extension on $(0,\infty)$. The 1^{st} quadrant corresponds to the intervals where $|u|$ is decreasing, which can be of the following types $(0,\infty)$, $(0,R_2)$,(R_1,∞),(R_1,R_2), $0 < R_1 < R_2 < \infty$. The 3^{rd} quadrant corresponds to the intervals (R_1,R_2) where $|u|$ is increasing.

The equation with absorption ($\varepsilon = -1$) is associated to the 2^{nd} and 4^{th} quadrant. It is known that the solutions have at most one zero, and their maximal interval of existence can be $(0,R_2)$, (R_1,∞), (R_1,R_2) or $(0,\infty)$. The 2^{nd} quadrant corresponds to the intervals (R_1,R_2) where $|u|$ is increasing. The 4^{th} quadrant corresponds to the intervals $(0,R_2)$ or (R_1,∞) where $|u|$ is decreasing.

The fixed points of (M_{scal}) are

$$M_0 = (X_0,,Z_0) = (\gamma, N-p-(p-1)\gamma), \quad (0,0), \quad N_0 = (0,N+a), \quad A_0 = (\frac{N-p}{p-1},0).$$

In particular M_0 is in the 1^{st} quadrant whenever $\gamma < \frac{N-p}{p-1}$, equivalently $Q > Q_1$, and in the 4^{th} quadrant whenever $Q < Q_1$. It corresponds to the solution

$$u(r) = Ar^{-\gamma}, \qquad \text{for } \varepsilon = 1, Q > Q_1, \qquad \text{or } \varepsilon = -1, Q < Q_1,$$

where $A = \left(\varepsilon\gamma^{p-1}(N-p-\gamma(p-1))\right)^{1/(Q-p+1)}$.

3.2 Local study

We examine the fixed points, where for simplicity we suppose $Q \neq Q_1$, and we deduce local results for the two equations:

• Point $(0,0)$: it is a saddle point, and the only trajectories that converge to $(0,0)$ are the separatrix, contained in the lines $X = 0, Y = 0$, they are not admissible.

• Point N_0 : it is a saddle point: the eigenvalues of the linearized system are $\frac{p}{p-1}$ and $-N$. the trajectories ending at N_0 at ∞ are located on the set $Z = 0$, then there exists a unique trajectory starting from $-\infty$ at N_0; it corresponds to the local existence and uniqueness of regular solutions, which we obtain easily.

• Point A_0 : the eigenvalues of the linearized system are $\frac{N-p}{p-1}$ and $\frac{N-p}{p-1}(Q_1 - Q)$. If $Q < Q_1$, A_0 is an unstable node. There is an infinity of trajectories starting from A_0 at $-\infty$; then $X(t)$ converges exponentially to $\frac{N-p}{p-1}$, thus $\lim_{r\to 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$. The corresponding solutions u satisfy the equation with a Dirac mass at 0. There exists no solution converging to A_0 at ∞. If $Q > Q_1$, A_0 is a saddle point; the trajectories starting from A_0 at $-\infty$ are not admissible; there is a trajectory converging at ∞, and then $\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u = \alpha > 0$.

• Point M_0 : the eigenvalues λ_1, λ_2 of the linearized system are the roots of equation

$$\lambda^2 + (Z_0 - X_0)\lambda + \frac{Q-p+1}{p-1} X_0 Z_0 = 0.$$

For $\varepsilon = 1$, M_0 is defined for $Q > Q_1$; the eigenvalues are imaginary when $X_0 = Z_0$, equivalently $\gamma = (N-p)/p$, $Q = Q_2$. When $Q < Q_2$, M_0 is a source, there exists an infinity of trajectories such that $\lim_{r\to 0} r^\gamma u = A$. When $Q > Q_2$, M_0 is a sink, and there exists an infinity of trajectories such that $\lim_{r\to\infty} r^\gamma u = A$. When $Q = Q_2$, M_0 is a center, from [2] For $\varepsilon = -1$, M_0 is defined for $Q < Q_1$, it is a saddle-point. There exist two trajectories $\mathcal{T}_1, \mathcal{T}_1'$ converging at ∞, such that $\lim_{r\to\infty} r^\gamma u = A$ and two trajectories $\mathcal{T}_2$, $\mathcal{T}_2'$, converging at 0, such that $\lim_{r\to 0} r^\gamma u = A$.

3.3 Global study

Remark 3.1 *System (M_{scal}) has no limit cycle for $Q \neq Q_2$. It is evident when $\varepsilon = -1$. When $\varepsilon = 1$, as noticed in [19], it comes from the Dulac's theorem: setting $X_t = f(X,Z)$, $\quad Z_t = g(X,Z)$, and*

$$B(X,Z) = X^{pQ/(Q+1-p)-2} Z^{(p/(Q+1-p)-1}, \quad M = B_X X_t + B_Z Z_t + B(f_X + g_Z),$$

then $M = KB$ with $K = (Q_2 - Q)\gamma(N-p)/p$, thus M has no zero for $Q \neq Q_2$.

Then from the Poincaré-Bendixson theorem, any trajectory bounded near $\pm\infty$ converges to one of the fixed points. Thus we find again global results:

• Equation with source ($\varepsilon = 1$). If $Q < Q_1$, there is no G.S.: the regular trajectory $\mathcal{T}$ issued from N_0 cannot converge to a fixed point. Then X tends to ∞ and the regular solutions u are changing sign, there is no G.S..

If $Q_1 < Q < Q_2$, the regular trajectory $\mathcal{T}$ cannot converge to M_0; if it converges to A_0, it is the unique trajectory converging to A_0; the set delimitated by $\mathcal{T}$ and $X = 0, Z = 0$ is invariant, thus it contains M_0; and the trajectories issued from M_0 cannot converge to a fixed point, which is contradictory. then again X tends to ∞ on $\mathcal{T}$ and the regular solutions u are changing sign.. The trajectory ending at A_0 converges to M_0 at $-\infty$; then there exist solutions $u > 0$ such that $\lim_{r\to 0} r^{\gamma}u = A$ and $\lim_{r\to 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$.

If $Q > Q_2$, the only singular solution at 0 is u_0, and the regular solutions are G.S., with $\lim_{r\to\infty} r^{\gamma}u = A$. Indeed M_0 is a sink; the trajectory ending at A_0 cannot converge to N_0 at $-\infty$, thus X converges to 0, and Z converges to ∞, then u cannot be positive on $(0,\infty)$.The trajectory issued from N_0 converges to M_0.

• Equation with absorption ($\varepsilon = -1$). If $Q > Q_1$, all the solutions u defined near 0 are regular; indeed the trajectories cannot converge to a fixed point.

If $Q < Q_1$, we find again easily a well known result: there exists a positive solution u_1, unique up to a scaling, such that $\lim_{r\to 0} r^{\frac{N-p}{p-1}} u_1 = \alpha > 0$, and $\lim_{r\to\infty} r^{\gamma}u_1 = A$. Indeed the eigenvalues at M_0 satisfy $\lambda_1 < 0 < \lambda_2$. There are two trajectories $\mathcal{T}_1, \mathcal{T}_1'$ associated to λ_1, and the eigenvector $(X_0 + |\lambda_1|, -\frac{X_0}{p-1})$. The trajectory $\mathcal{T}_1$ satisfies $X_t > 0 > Z_t$ near ∞, and $X > \frac{N-p}{p-1}$, since $Z_0 < 0$, and X cannot take the value $\frac{N-p}{p-1}$ because at such a point $X_t < 0$; then $\frac{N-p}{p-1} < X < X_0$ and $X_t > 0$ as long as it is defined; similarly $Z_0 < Z < 0$ and $Z_t < 0$; then $\mathcal{T}_1$ converge to a fixed point, necessarily A_0, showing the existence of u_1. The trajectory $\mathcal{T}_1'$ corresponds to solutions u such that $\lim_{r\to\infty} r^{\gamma}u = A$ and $\lim_{r\to R} u = \infty$ for some $R > 0$. There are two trajectories $\mathcal{T}_2$, $\mathcal{T}_2'$, associated to λ_2, defining solutions u such that $\lim_{r\to 0} r^{\gamma}u = A$ and changing sign, or with a minimum point and $\lim_{r\to R} u = \infty$ for some $R > 0$. The regular trajectory starts from N_0 in the 2^{nd} quadrant, it cannot converge to a fixed point, then $\lim_{r\to R} u = \infty$ for some $R > 0$.

• Critical case $Q = Q_2$: it is remarkable that system (M_{scal}) admits <u>*another invariant line*</u>, namely A_0N_0, given by

$$\frac{X}{p'} + \frac{Z}{Q_2+1} - \frac{N-p}{p} = 0. \tag{3.3}$$

It precisely corresponds to well-known solutions of the two equations

$$u = c(K^2 + r^{(p+a)/(p-1)})^{(p-N)/(p+a)}, \text{for } \varepsilon = 1; \quad u = c\left|K^2 - r^{(p+a)/(p-1)}\right|^{(p-N)/(p+a)}, \text{for } \varepsilon = -1,$$

where $K^2 = c^{Q-p+1}(N+a)^{-1}\left((N-p)/(p-1)\right)^{1-p}$.

Remark 3.2 *The global results have been obtained <u>without using energy functions</u>. The study of [2] was based on a reduction of type of Remark 2.2, using an energy function linked to the new unknown. Other energy functions are well-known, of Pohozaev type:*

$$\mathcal{F}_\sigma(r) = r^N\left[\frac{|u'|^p}{p'} + \varepsilon r^a\frac{|u|^{Q+1}}{Q+1} + \sigma\frac{u\,|u'|^{p-2}\,u'}{r}\right] = r^{N-p}\,|u|^p\,|X|^{p-2}\,X\left[\frac{X}{p'} + \frac{Z}{Q+1} - \sigma\right],$$

with $\sigma = \frac{N-p}{p}$, *satisfying* $\mathcal{F}'_\sigma(r) = r^{N-1+a}\left(\frac{N+a}{Q+1} - \frac{N-p}{p}\right)|u|^{Q+1}$, *or with* $\sigma = \frac{N+a}{Q+1}$, *leading to* $\mathcal{F}'_\sigma(r) = r^{N-1}\left(\frac{N+a}{Q+1} - \frac{N-p}{p}\right)|u'|^p$. *In the critical case* $Q = Q_2$, *all these functions coincide and they are constant, in other words system* (M_{scal}) *has a first integral. We find again the line (3.3): the G.S. are the functions of energy* 0.

4 Local study of system (S)

In all the sequel we study the system with source terms: $(G) = (S)$. Assumption (1.5) is the most interesting case for studying the existence of the G.S.

We first study the local behaviour of nonnegative solutions (u,v) defined near 0 or near ∞. It is well known that any solution (u,v) positive on some interval $(0,R)$ satisfies $u',v' < 0$ on $(0,R)$. Any solution (u,v) positive on (R,∞), satisfies $u',v' < 0$ near ∞. We are reduced to study the system in the region $\mathcal{R}$ where $X,Y,Z,W > 0$, and consider the fixed points in $\bar{\mathcal{R}}$. Then

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = \frac{r^{1+a}u^s v^\delta}{|u'|^{p-1}}, \quad W(t) = \frac{r^{1+b}v^m u^\mu}{|v'|^{q-1}}; \tag{4.1}$$

and (X,Y,Z,W) is a solution of system (M) in $\mathcal{R}$ if and only if (u,v) defined by

$$u=r^{-\gamma}(ZX^{p-1})^{(q-1-m)/D}(WY^{q-1})^{\delta/D}, \qquad v=r^{-\xi}(WY^{q-1})^{(p-1-s)/D}(ZX^{p-1})^{\mu/D} \tag{4.2}$$

is a positive solution with $u',v' < 0$. Among the fixed points, the point M_0 defined at (2.6) lies in $\mathcal{R}$ if and only if

$$0 < \gamma < \frac{N-p}{p-1} \quad \text{and} \quad 0 < \xi < \frac{N-q}{q-1}. \tag{4.3}$$

The local study of the system near M_0 appears to be tricky, see Remark 4.2. A main difference with the scalar case is that there always exist a trajectory converging to M_0 at $\pm\infty$:

Proposition 4.1 *(Point* M_0*) Assume that (4.3) holds. Then there exist trajectories converging to* M_0 *as* $r \to \infty$, *and then solutions* (u,v) *being defined near* ∞, *such that*

$$\lim_{r\to\infty} r^\gamma u = \alpha > 0, \quad \lim_{r\to\infty} r^\xi v = \beta > 0. \tag{4.4}$$

There exist trajectories converging to M_0 *as* $r \to 0$, *and thus solutions* (u,v) *being defined near* 0 *such that*

$$\lim_{r\to 0} r^\gamma u = \alpha > 0, \quad \lim_{r\to 0} r^\xi v = \beta > 0. \tag{4.5}$$

Proof. Here $M_0 \in \mathcal{R}$; setting $X = X_0 + \tilde{X}, Y = Y_0 + \tilde{Y}, Z = Z_0 + \tilde{Z}, W = W_0 + \tilde{W}$, the linearized system is

$$\begin{cases} \tilde{X}_t = X_0(\tilde{X} + \frac{1}{p-1}\tilde{Z}), \\ \tilde{Y}_t = Y_0(\tilde{Y} + \frac{1}{q-1}\tilde{W}), \\ \tilde{Z}_t = Z_0(-s\tilde{X} - \delta\tilde{Y} - \tilde{Z}), \\ \tilde{W}_t = W_0(-\mu\tilde{X} - m\tilde{Y} - \tilde{W}). \end{cases}$$

The eigenvalues are the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, of equation

$$\left[(\lambda - X_0)(\lambda + Z_0) + \frac{s}{p-1}X_0Z_0\right]\left[(\lambda - Y_0)(\lambda + W_0) + \frac{m}{q-1}Y_0W_0\right] - \frac{\delta\mu}{(p-1)(q-1)}X_0Y_0Z_0W_0 = 0. \tag{4.6}$$

This equation is of the form

$$f(\lambda) = \lambda^4 + E\lambda^3 + F\lambda^2 + G\lambda - H = 0,$$

with

$$\begin{cases} E = Z_0 - X_0 + W_0 - Y_0, \\ F = (Z_0 - X_0)(W_0 - Y_0) - \frac{s+p-1}{p-1}X_0Z_0 - \frac{m+q-1}{q-1}Y_0W_0, \\ G = -\frac{q-1-m}{q-1}Y_0W_0(Z_0 - X_0) - \frac{p-1-s}{p-1}X_0Z_0(W_0 - Y_0), \\ H = \frac{D}{(p-1)(q-1)}X_0Y_0Z_0W_0. \end{cases}$$

From (1.5) we have $H > 0$, then $\lambda_1\lambda_2\lambda_3\lambda_4 < 0$. There exist two real roots $\lambda_3 < 0 < \lambda_4$, and two roots λ_1, λ_2, real with $\lambda_1\lambda_2 > 0$, or complex. Therefore there exists at least one trajectory converging to M_0 at ∞ and another one at $-\infty$. Then (4.4) and (4.5) follow from (4.2). Moreover the convergence is monotone for X, Y, Z, W. ■

Remark 4.2 *There exist imaginary roots, namely* $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = 0$, *if and only if there exists* $c > 0$ *such that* $f(ci) = 0$, *that means* $Ec^2 - G = 0$, *and* $c^4 - Fc^2 - H = 0$, *equivalently*

$$E = G = 0, \quad \text{or } EG > 0 \text{ and } G^2 - EFG - E^2H = 0.$$

Condition $E = G = 0$ *means that*

(i) either $Z_0 = X_0$ *and* $W_0 = Y_0$, *i.e.*

$$(\gamma, \xi) = (\frac{N-p}{p}, \frac{N-q}{q}), \tag{4.7}$$

in other words $(\delta, \mu) = (\frac{q(N(p-1-s)+p(1-s+a))}{p(N-q)}, \frac{p(N(q-1-m)+q(1-m+b))}{q(N-p)})$.

(ii) or $(p-1-s)(q-1-m) > 0$ *and* (γ, ξ) *satisfies*

$$\begin{cases} 2N - p - q = p\gamma + q\xi, \\ (1 - \frac{m}{q-1})\xi(N - q - (q-1)\xi) = (1 - \frac{s}{p-1})\gamma(N - p - (p-1)\gamma). \end{cases} \tag{4.8}$$

This gives in general 0,1 or 2 values of (γ, ξ). *For example, in the case* $\frac{m}{q-1} = \frac{s}{p-1} \neq 1$, *and* $(p-2)(q-2) > 0$ *and* $N > \frac{pq-p-q}{p+q-2}$ *we find another value, different from the one of (4.7) for* $p \neq q$:

$$(\gamma, \xi) = (N\frac{q-2}{pq-p-q} - 1, N\frac{p-2}{pq-p-q} - 1). \tag{4.9}$$

Moreover the computation shows that it can exist imaginary roots with $E, G \neq 0$.

In the case $p = q = 2$ and $s = m$ the situation is interesting:

Proposition 4.3 *Assume $p = q = 2$ and $s = m < \frac{N}{N-2}$, with $\delta + 1 - s > 0, \mu + 1 - s > 0$. In the plane (δ, μ), let $\mathcal{H}_s$ be the hyperbola of equation*

$$\frac{1}{\delta + 1 - s} + \frac{1}{\mu + 1 - s} = \frac{N-2}{N-(N-2)s}, \tag{4.10}$$

equivalently $\gamma + \xi = N - 2$. Then $\mathcal{H}_s$ is contained in the set of points (δ, μ) for which the linearized system at M_0 has imaginary roots, and equal when $s \leqq 1$.

Proof. The assumption $D > 0$ imply $\delta + 1 - s > 0$ and $\mu + 1 - s > 0$; condition $E = G = 0$ implies $s < N/(N-2)$ and reduces to condition (4.10). Moreover if $s \leqq 1$, all the cases are covered. Indeed $2G = (s-1)E\left[Y_0 Z_0 + X_0 W_0\right]$, hence $GE \leqq 0$. ■

Next we give a summary of the local existence results obtained by linearization around the other fixed points of system (M) proved in Section 10. Recall that $t \to -\infty$ as $r \to 0$ and $t \to \infty$ as $r \to \infty$.

Proposition 4.4 *(Point N_0) A solution (u, v) is regular if and only if the corresponding trajectory converges to N_0 when $r \to 0$. For any $u_0, v_0 > 0$, there exists a unique local regular solution (u, v) with initial data (u_0, v_0).*

Proposition 4.5 *(Point A_0) If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N + a$ and $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N + b$, there exist (admissible) trajectories converging to A_0 when $r \to \infty$. If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N + a$ and $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N + b$, the same happens when $r \to 0$. In any case*

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^{\frac{N-q}{q-1}} v = \beta > 0. \tag{4.11}$$

If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N + a$ or $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N + b$, there exists no trajectory converging when $r \to \infty$; if $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N + a$ or $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N + b$, there exists no trajectory converging when $r \to 0$.

Proposition 4.6 *(Point P_0) 1) Assume that $q > m + 1$ and $q + b < \frac{N-p}{p-1}\mu < N + b - m\frac{N-q}{q-1}$. If $\gamma < \frac{N-p}{p-1}$ there exist trajectories converging to P_0 when $r \to \infty$ (and not when $r \to 0$). If $\gamma > \frac{N-p}{p-1}$ the same happens when $r \to 0$ (and not when $r \to \infty$).*

2) Assume that $q < m + 1$ and $q + b > \frac{N-p}{p-1}\mu > N + b - m\frac{N-q}{q-1}$ and $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1) + (b+1)q$. If $\gamma < \frac{N-p}{p-1}$ there exist trajectories converging to P_0 when $r \to 0$ (and not when $r \to \infty$). If $\gamma > \frac{N-p}{p-1}$ there exist trajectories converging when $r \to \infty$ (and not when when $r \to 0$).

In any case, setting $\kappa = \frac{1}{q-1-m}(\frac{N-p}{p-1}\mu - (q+b))$, there holds

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^{\kappa} v = \beta > 0. \tag{4.12}$$

Remark 4.7 *This result improves the results of existence obtained by the fixed point theorem in [27] in the case of system (RP) with $p = q = 2, a = 0, N = 3$, $2s + m \neq 3$. The proof is quite simpler..*

Proposition 4.8 *(Point I_0) If $\frac{N-p}{p-1}s > N+a$ and $\frac{N-q}{q-1}\mu > N+b$, there exist trajectories converging to I_0 when $r \to \infty$, and then*

$$\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u = \beta, \quad \lim_{r\to\infty} v = \alpha > 0. \tag{4.13}$$

For any $s, m \geqq 0$, there is no trajectory converging when $r \to 0$.

Proposition 4.9 *(Point G_0) Suppose $\frac{N-p}{p-1}\mu < N+b$. If $q+b < \frac{N-p}{p-1}\mu$ and $N+a < \frac{N-p}{p-1}s$, there exist trajectories converging to G_0 when $r \to \infty$. If $\frac{N-p}{p-1}\mu < q+b$ and $\frac{N-p}{p-1}s < N+a$, the same happens when $r \to 0$. In any case*

$$\lim r^{\frac{N-p}{p-1}} u = \beta, \quad \lim v = \alpha > 0. \tag{4.14}$$

Proposition 4.10 *(Point C_0) Suppose $N+b < \frac{N-q}{q-1}m$ (hence $q < m+1$) with $m \neq \frac{N(q-1)+(b+1)q}{N-q}$, and $\delta > \frac{(N+a)(m+1-q)}{q+b}$. Then there exist trajectories converging to C_0 when $r \to \infty$ (and not when $r \to 0$), and then*

$$\lim u = \alpha > 0, \quad \lim r^k v = \beta, \tag{4.15}$$

where $k = \frac{q+b}{m+1-q}$.

Proposition 4.11 *(Point R_0) Assume that $N+b < \frac{N-q}{q-1}m$ (hence $q < m+1$) with $m \neq \frac{N(q-1)+b+bq}{N-q}$, and $\delta < \frac{(N+a)(m+1-q)}{q+b}$. If $\frac{(p+a)(m+1-q)}{q+b} < \delta$, there exist trajectories converging to R_0 when $r \to \infty$ (and not when $r \to 0$). If $\delta < \frac{(p+a)(m+1-q)}{q+b}$, there exist trajectories converging when $r \to 0$ (and not when $r \to \infty$), and then (4.15) holds again.*

We obtain similar results of convergence to the points Q_0, J_0, H_0, D_0, S_0 by exchanging p, δ, s, a and q, μ, m, b. There is no admissible trajectory converginf to $0, K_0, L_0$, see Remark 10.1.

5 Global results for system (S)

We are concerned by the existence of global positive solutions. First we find again easily some known results by using our dynamical approach.

Proposition 5.1 *Assume that system (S) admits a positive solution (u,v) in $(0,\infty)$. Then the corresponding solution (X,Y,Z,W) of system (M) stays in the box*

$$\mathcal{A} = \left(0, \frac{N-p}{p-1}\right) \times \left(0, \frac{N-q}{q-1}\right) \times (0, N+a) \times (0, N+b), \tag{5.1}$$

in other words

$$ru' + \frac{N-p}{p-1}u > 0, \quad rv' + \frac{N-q}{q-1}v > 0, \quad r^{1+a}u^s v^\delta < (N+a)\left|u'\right|^{p-1}, \quad r^{1+b}u^\mu v^m < (N+b)\left|v'\right|^{q-1}. \tag{5.2}$$

and then

$$u^{s-p+1}v^{\delta} \leqq C_1 r^{-(p+a)}, \quad u^{\mu}v^{m-q+1} \leqq C_2 r^{-(q+b)}, \quad \text{in } (0,\infty), \tag{5.3}$$

where $C_1 = (N+a)(\frac{N-p}{p-1})^{p-1}, C_2 = (N+b)(\frac{N-q}{q-1})^{q-1}$, *and*

$$\lim_{r\to 0} r^{\frac{N-p}{p-1}} u = c_1 \geqq 0, \quad \lim_{r\to 0} r^{\frac{N-q}{q-1}} v(r) = c_2 \geqq 0, \quad \lim\inf_{r\to\infty} r^{\frac{N-p}{p-1}} u > 0, \quad \lim\inf_{r\to\infty} r^{\frac{N-q}{q-1}} v > 0. \tag{5.4}$$

As a consequence if $s \leqq p-1$ *or* $m \leqq q-1$, *we have*

$$u \leqq K_1 r^{-\gamma}, \quad v \leqq K_2 r^{-\xi}, \quad \text{in } (0,\infty), \tag{5.5}$$

with $K_1 = C_1^{(q-1-m)/D} C_2^{\delta/D}, K_2 = C_1^{\mu/D} C_2^{(p-1-s)/D}$.

Proof. The solution of system (M) in $\mathcal{R}$ defined on $\mathbb{R}$. On the hyperplane $X = \frac{N-p}{p-1}$ we have $X_t > 0$, the field is going out. If at some time t_0, $X(t_0) = \frac{N-p}{p-1}$, then $X(t) > \frac{N-p}{p-1}$ for $t > t_0$, in turn $X_t \geqq X\left[X - \frac{N-p}{p-1}\right] > 0$, since since $Z > 0$, thus $X(t) > 2\frac{N-p}{p-1}$ for $t > t_1 > t_0$, then $X_t \geqq X^2/2$, which implies that X blows up in finite time; thus $X(t) < \frac{N-p}{p-1}$ on $\mathbb{R}$; in the same way $Y(t) < \frac{N-q}{q-1}$. On the hyperplane $Z = N+a$ we have $Z_t < 0$, the field is entering. If at some time t_0, $Z(t_0) = N+a$ then $Z(t) > N+a$ for $t < t_0$, then $Z_t \leqq Z(N+a-Z)$, since $sX + \delta Y > 0$, and Z blows up in finite time as above; thus $Z(t) < N+a$ on $\mathbb{R}$, in the same way $W(t) < N+b$. Then (5.2),(5.3) and (5.5) follows. By integration it implies that $r^{(N-p)/(p-1)}u(r)$ is nondecreasing near 0 or ∞, hence (5.4) holds. ■

Next we prove Theorem 1.1.

Proof of Theorem 1.1. (i) The trajectories of the regular solutions start from $N_0 = (0,0,N+a,N+b)$, from Proposition 4.4, and the unstable variety $\mathcal{V}_u$ has dimension 2, from (10.1), (10.2). It is given locally by $Z = \varphi(X,Y), W = \psi(X,Y)$ for $(X,Y) \in B(0,\rho)\backslash\{0\} \subset \mathbb{R}^2$.

To any $(x,y) \in B(0,\rho)\backslash\{0\}$ we associate the unique trajectory $\mathcal{T}_{x,y}$ in $\mathcal{V}_u$ going through this point. If T^* is the maximal interval of existence of a solution on $\mathcal{T}_{x,y}$, then $\lim_{t\to T^*}(X(t)+Y(t)) = \infty$. Indeed Z, and W satisfy $0 < Z < N+a$, $0 < W < N+b$ as long as the solution exists, because at a time T where $Z(T) = N+a$, we have $Z_t < 0$. If there exists a first time T such that $X(T) = \frac{N-p}{p-1}$ or $Y(T) = \frac{N-q}{q-1}$, then $T < T^*$. We consider the open rectangle $\mathcal{N}$ of submits

$$(0,0), \quad \varpi_1 = \left(\frac{N-p}{p-1}, 0\right), \quad \varpi_2 = \left(0, \frac{N-q}{q-1}\right), \quad \varpi = \left(\frac{N-p}{p-1}, \frac{N-q}{q-1}\right).$$

Let $\mathcal{U} = \{(x,y) \in B(0,\rho) : x,y > 0\}$; then $\mathcal{U} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}$, where

$$\begin{cases} \mathcal{S}_i = \{(x,y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ on } (\varpi_i, \varpi)\}, \quad i = 1,2, \\ \mathcal{S}_3 = \{(x,y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ at } \varpi\}, \quad \mathcal{S} = \{(x,y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ stays in } \mathcal{N}\}. \end{cases}$$

Any element of $\mathcal{S}$ defines a G.S. Assume $s < \frac{N(p-1)+p+pa}{N-p}$. Let us show that $\mathcal{S}_1$ is nonempty. Consider the trajectory $\mathcal{T}_{\bar{x},0}$ on $\mathcal{V}_u$ associated to $(\bar{x},0)$, with $\bar{x} \in (0,\rho)$, going through $\bar{M} = (\bar{x}, 0,$

$\varphi(\bar{x},0),\psi(\bar{x},0))$; it is <u>*not admissible*</u> for our problem, since it is in the hyperplane $Y=0$: it satisfies the system

$$\left\{\begin{array}{c} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Z_t = Z\left[N + a - sX - Z\right], \\ W_t = W\left[N + b - \mu X - W\right], \end{array}\right.$$

which is <u>*not completely coupled*</u>. The two equations in X, Z corresponds to the equation

$$-\Delta_p U = r^a U^s. \tag{5.6}$$

The regular solutions of (5.6) are changing sign, since s is subcritical, see Section 3. Consider the solution $(\bar{X},\bar{Y},\bar{Z},\bar{W})$ of system (M), of trajectory $\mathcal{T}_{\bar{x},0}$, going through $\bar{M}$ at time 0; it satisfies $\bar{Y}=0$, and $\bar{X}(t)>0$, $\bar{Z}(t)>0$ tend to ∞ in finite time T^*, then for any given $C \geqq \frac{N-p}{p-1}$, there exist a first time $T<T^*$ such that $\bar{X}(T)=C$, and $\bar{Y}(T)=0$. We have $\lim_{t\to-\infty}\bar{W}=N+b$, and necessarily $0<\bar{W}<N+b$, in particular $0<\bar{W}(T)<N+b$; and $\bar{W}_t$ is bounded on $(-\infty,T^*)$, then $\bar{W}$ has a finite limit at T^*. The field at time T is transverse to the hyperplane $X=\frac{N-p}{p-1}$: we have $\bar{X}_t \geqq C\frac{Z(T)}{p-1}>0$, since $\bar{Z}(T)>0$. From the continuous dependance of the initial data at time 0, for any $\varepsilon>0$, there exists $\eta>0$ such that for any $(x,y)\in B((\bar{x},0),\eta)$ and for any (X,Y,Z,W) on $\mathcal{T}_{x,y}$, there exists a first time T_ε such that $X(T_\varepsilon)=C$, and $|Y(t)|\leqq\varepsilon$ for any $t\leqq T_\varepsilon$, in particular for any $(x,y)\in B((\bar{x},0),\eta)$ with $y>0$, and then $0<Y(t)\leqq\varepsilon$ for any $t\leqq T_\varepsilon$. Let us take $C=\frac{N-p}{p-1}$. Then $(x,y)\in\mathcal{S}_1$. The same arguments imply that $\mathcal{S}_1$ is open. Similarly assuming $m<\frac{N(q-1)+q+qb}{N-q}$ implies that $\mathcal{S}_2$ is nonempty and open. By connexity $\mathcal{S}$ is empty if and only if $\mathcal{S}_3$ is nonempty.

(ii) Here the difficulty is due to the fact that the zeros of u,v correspond to infinite limits for X,Y, and then the argument of continuous dependance is no more available. We can write $\mathcal{U}=\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3\cup\mathcal{S}$, where

$$\left\{\begin{array}{c} \mathcal{M}_1 = \{(x,y)\in\mathcal{U} \text{ and } \mathcal{T}_{x,y} \text{ has an infinite branch in } X \text{ with } Y \text{ bounded}\}, \\ \mathcal{M}_2 = \{(x,y)\in\mathcal{U}: \ \mathcal{T}_{x,y} \text{ has an infinite branch in } Y \text{ with } X \text{ bounded}\}, \\ \mathcal{M}_3 = \{(x,y)\in\mathcal{U}: \ \mathcal{T}_{x,y} \text{ has an infinite branch in } (X,Y)\}. \end{array}\right.$$

In other words, $\mathcal{M}_1$ is the set of $(x,y)\in\mathcal{U}$ such that for any (X,Y,Z,W) on $\mathcal{T}_{x,y}$, there exists a T^* such that $\lim_{t\to T^*}X(t))=\infty$, and $Y(t)$ stays bounded on $(-\infty,T^*)$, that means the set of $(x,y)\in\mathcal{U}$ such that for any solution (u,v) corresponding to $\mathcal{T}_{x,y}$, u vanishes before v; similarly for $\mathcal{M}_2$. Otherwise $\mathcal{M}_3$ is the set of $(x,y)\in\mathcal{U}$ such that there exists a T^* such that $\lim_{t\to T^*}X(t)=\lim_{t\to T^*}Y(t)=\infty$, that means (u,v) vanish at the same $R^*=e^{T^*}$. In that case, from the Höpf Lemma, $\lim_{r\to R}\frac{u'}{(r-R)u}=1$, then $\lim_{t\to T^*}\frac{X}{Y}=1$.

We are lead to show that $\mathcal{M}_1$ is nonempty and open for $s<\frac{N(p-1)+p+pa}{N-p}$. We consider again the trajectory $\bar{\mathcal{T}}$ and take C large enough: $C=2(\frac{N-p}{p-1}+\frac{N+|b|}{q-1})$. Let $\varepsilon\in\left(0,\frac{C}{2}\right)$. For any $(x,y)\in B((\bar{x},0),\eta)$ with $y>0$, and any (X,Y,Z,W) on $\mathcal{T}_{x,y}$, there is a first time T_ε such that $X(T_\varepsilon)=C$, and $0<Y(t)\leqq\varepsilon$ for any $t\leqq T_\varepsilon$. And X is increasing and $X_t\geqq X(X-C)$, thus there exists T^* such that $\lim_{t\to T^*}X(t)=\infty$. Setting $\varphi=X/Y$, we find

$$\frac{\varphi_t}{\varphi} = X - Y + \frac{Z}{p-1} - \frac{W}{q-1} + \frac{N-q}{q-1} - \frac{N-p}{p-1} \geqq X - Y - \frac{C}{2}$$

then $\varphi_t(T_\varepsilon) > 0$. Let $\theta = \sup\{t > T_\varepsilon : \varphi_t > 0\}$; suppose that θ is finite; then $\varphi(\theta) > \varphi(T_\varepsilon) = C/\varepsilon > 2$ and $X(\theta) \leqq Y(\theta) + C < X(\theta)/2 + C$, which is contradictory. Then φ is increasing up to T^*; if $\lim_{t\to T^*} Y(t) = \infty$, then $\lim_{t\to T^*} \varphi = 1$, which is impossible. Then $(x, y) \in \mathcal{M}_1$, thus $\mathcal{M}_1$ is nonempty. In the same way $\mathcal{M}_1$ is open. Indeed for any $(\bar{x}, \bar{y}) \in \mathcal{M}_1$ there exists $M > 0$ such that $0 < \bar{Y}(t) \leqq M/2$ on $\mathcal{T}_{\bar{x},\bar{y}}$. To conclude we argue as above, with $(\bar{x}, 0)$ replaced by $(\bar{x}, \bar{y})$, and C replaced by $C + M$. ■

Proof of Proposition 1.2. Assume $s \geqq \frac{N(p-1)+p+pa}{N-p}$. Consider the Pohozaev type function

$$\mathcal{F}(r) = r^N\left[\frac{|u'|^p}{p'} + \frac{r^a u^{s+1}}{s+1}v^\delta + \frac{N-p}{p}\frac{u\,|u'|^{p-2}u'}{r}\right] = r^{N-p}u^p\left[\frac{X}{p'} + \frac{1}{s+1}Z - \frac{N-p}{p}\right]. \tag{5.7}$$

We find $\mathcal{F}(0) = 0$ and

$$\begin{aligned}\mathcal{F}'(r) &= r^{N-1+a}\left[\left(\frac{N+a}{s+1} - \frac{N-p}{p}\right)v^\delta u^{s+1} + \frac{\delta}{s+1}ru^{s+1}v^{\delta-1}v'\right]\\ &= r^{N-1+a}v^\delta u^{s+1}\left[\frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1}\right]\end{aligned} \tag{5.8}$$

From our assumption, $\mathcal{F}$ is decreasing, and $Z > 0$, thus $X < \frac{N-p}{p-1}$. Then $\mathcal{S}_1, \mathcal{S}_3$ are empty. If moreover $m \geqq \frac{N(q-1)+q+qb}{N-q}$ then $\mathcal{S}_2$ is empty, therefore $\mathcal{S} = \mathcal{U}$. ■

Remark 5.2 *Let us only assume that $s \geqq \frac{N(p-1)+p+pa}{N-p}$. If one function has a first zero, it is v. Indeed if there exists a first value R where $u(R) = 0$, and $v(r) > 0$ on $[0, R)$, then $\mathcal{F}(R) = \frac{R^N}{p'}|u'(R)|^p > 0$.*

As a first consequence we obtain existence results for the Dirichlet problem. It solves an open problem in the case $s > p-1$ or $m > q-1$, and extends some former results of [12] and [42]. Our proof, based on the shooting method differs from the proof of [12], based on degree theory and blow-up technique. Our results extend the ones of [3, Theorem 2.2] relative to the case $p = q = 2$, obtained by studying the equation satisfied by a suitable function of u, v.

Corollary 5.3 *system (S) admits no G.S. and then there is a radial solution of the Dirichlet problem in a ball in any of the following cases:*

(i) $p < s+1, q < m+1$, *and* $\min(s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a), \frac{N-p}{p-1}\mu + m\frac{N-q}{q-1} - (N+b)) \leqq 0$;

(ii) $p < s+1$, $q > m+1$ *and* $s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a) \leqq 0$ *or* $\gamma - \frac{N-p}{p-1} > 0$;

(iii) $p > s+1, q > m+1$ *and* $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) \geqq 0$;

(iv) $p \geqq s+1, q \geqq m+1$ *and* $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) > 0$.

Proof. From Theorem 1.1, we are reduced to prove the nonexistence of G.S.

(i) Assume $p < s+1$, and $s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a) < 0$. We have $-\Delta_p u \geqq Cr^{a-\frac{N-q}{q-1}\delta}u^s$ for large r. From [6, Theorem 3.1], we find $u = O(r^{-(p+a-\frac{N-q}{q-1}\delta)/(s+1-p)})$, and then $s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a) \geqq 0$, from (5.4), which contradicts our assumption. In case of equality, we find $-\Delta_p u \geqq Cr^{-N}$ for large r, which is impossible. Then there exists no G.S. This improves ythe result of [12] where the minimum is replaced by a maximum.

(ii) Assume $p < s+1$, $q > m+1$ and $\gamma - \frac{N-p}{p-1} > 0$; then $u = O(r^{-\gamma})$, which contradicts (5.4). If $\gamma - \frac{N-p}{p-1} = 0$, then $\lim r^{\frac{N-p}{p-1}} u = \alpha > 0$, and $\xi > \frac{N-q}{q-1}$. Hence $-\Delta_q v \geqq Cr^{b-\frac{N-p}{p-1}\mu} v^m$ for large r, then $v \geqq Cr^{(q+b-\frac{N-p}{p-1}\delta)/(q-1-m)} = Cr^{-\xi}$. There exists $C_1 > 0$ such that $C_1 \leqq r^{\xi} v \leqq 2C_1$ for large r, from [6, Theorem 3.1] and (5.5), then $-\Delta_p u \geqq Cr^{-N}$ for some $C > 0$, which is again contradictory.

(iii) (iv) The nonexistence of G.S is obtained by extension of the proof of [12] to the case $a, b \neq 0$. Moreover (iii) implies the nonexistence of positive solution (u, v), radial or not, in any exterior domain $(R, \infty) \times (R, \infty), R > 0$ from [6]. ■

Corollary 5.4 *Assume (4.3) with $p = q = 2$. If $\delta + s \geqq \frac{N+2+2a}{N-2}$ and $\mu + m \geqq \frac{N+2+2b}{N-2}$, then system (S) admits a G.S.*

Proof. It was shown in [28], [41] by the moving spheres method that the Dirichlet problem has no radial or nonradial solution. Then Theorem 1.1 applies again. ■

We aso extend and improve a result of nonexistence of [10] for the case $p = q = 2, a = 0, s > 1$:

Proposition 5.5 *Assume $s + 1 > p$ or $\gamma > \frac{N-p}{p}$, and*

$$s + \frac{p(N-q)}{(q-1)(N-p)}\delta < \frac{N(p-1) + pa + p}{N-p} \tag{5.9}$$

Then system (S) admits no G.S. and then there is a solution of the Dirichlet problem. The same happens by exchanging p, s, δ, a, γ with q, m, μ, b, ξ.

Proof. Consider the function $\mathcal{F}$ defined at (5.7). Suppose that there exists a G.S. Then from (5.1) and (5.9) we find

$$\frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1} > \frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta}{s+1}\frac{N-q}{q-1} \geqq 0.$$

From (5.8), we deduce that $\mathcal{F}$ is nondecreasing. First suppose $s + 1 > p$. From (5.3) and (5.4),it follows that $u = O(r^{-k})$ at ∞, with $k = (p+a-\delta\frac{N-q}{q-1})/(s-p+1)$. In turn $r^{N-p}u^p = O(r^{(N-p)-kp}) = o(1)$ from (5.9), then $\mathcal{F}(r) = o(1)$ near ∞. Next assume $s + 1 \leqq p$ and $\gamma > \frac{N-p}{p}$. Then $r^{N-p}u^p = O(r^{N-p-\gamma p})$, hence $\mathcal{F}(r) = o(1)$ near ∞. In any case we get a contradiction. ■

6 The Hamiltonian system

Here we consider the nonnegative solutions of the variational Hamiltonian problem (SH) in $\Omega \subset \mathbb{R}^N$

$$(SH) \begin{cases} -\Delta u = |x|^a v^{\delta}, \\ -\Delta v = |x|^b u^{\mu}, \end{cases}$$

where $p=q=2<N$, $s=m=0$, $a>b>-2$, and $D=\delta\mu-1>0$. For this case we find

$$\gamma=\frac{(2+a)+(2+b)\delta}{D},\quad \xi=\frac{2+b+(2+a)\mu}{D},\quad \gamma+2+a=\delta\xi,\quad \xi+2+b=\mu\gamma.$$

The particular solution $(u_0(r),v_0(r))=(Ar^{-\gamma},Br^{-\xi})$ exists for $0<\gamma<N-2$, $0<\xi<N-2$. Here X,Y,Z,W are defined by

$$X(t)=\frac{r\,|u'|}{u},\qquad Y(t)=\frac{r\,|v'|}{v},\qquad Z(t)=\frac{r^{1+a}v^{\delta}}{|u'|},\qquad W(t)=\frac{r^{1+b}u^{\mu}}{|v'|},$$

with $t=\ln r$, and system (M) becomes

$$(MH)\left\{\begin{array}{l} X_t=X\left[X-(N-2)+Z\right],\\ Y_t=Y\left[Y-(N-2)+W\right],\\ Z_t=Z\left[N+a-\delta Y-Z\right],\\ W_t=W\left[N+b-\mu X-W\right]\end{array}\right.$$

This system has a <u>*Pohozaev type*</u> function, well known at least in the case $a=b=0$, given at (1.7):

$$\begin{aligned}\mathcal{E}_H(r)&=r^N\left[u'v'+r^b\frac{|u|^{\mu+1}}{\mu+1}+r^a\frac{|v|^{\delta+1}}{\delta+1}+\frac{N+a}{\delta+1}\frac{vu'}{r}+\frac{N+b}{\mu+1}\frac{uv'}{r}\right]\\ &=r^{N-2}uv\left[XY-\frac{Y(N+b-W)}{\mu+1}-\frac{(N+a-Z)X}{\delta+1}\right]\\ &=r^{N-2-\gamma-\xi}(ZX)^{(\mu+1)/D}(WY)^{(\delta+1)/D}\left[XY-\frac{Y(N+b-W)}{\mu+1}-\frac{(N+a-Z)X}{\delta+1}\right].\end{aligned}$$

It can also be found by a direct computation, and $\mathcal{E}_H$ satisfies

$$\mathcal{E}_H'(r)=r^{N-1}u'v'\left(\frac{N+a}{\delta+1}+\frac{N+b}{\mu+1}-(N-2)\right).$$

We define the <u>*critical case*</u> as the case where (δ,μ) lie on the hyperbola $\mathcal{H}_0$ given by

$$\frac{N+a}{\delta+1}+\frac{N+b}{\mu+1}=N-2,\ \text{equivalently}\ \gamma+\xi=N-2. \tag{6.1}$$

In this case $\gamma=\frac{N+b}{\mu+1},\xi=\frac{N+a}{\delta+1}$, and $\mathcal{E}_H'(r)\equiv 0$. It corresponds to the existence of a first integral of system (M), which can also be expressed in the variables $\mathrm{U}=r^{\gamma}u$, $\mathrm{V}=r^{\xi}v$ of Remark 2.2:

$$\mathcal{E}_H(r)=\mathrm{U}_t\mathrm{V}_t-\gamma\xi\mathrm{UV}+\frac{\mathrm{U}^{\mu+1}}{\mu+1}+\frac{\mathrm{V}^{\delta+1}}{\delta+1}=C.$$

The supercritical case is defined as the case where (δ,μ) is above $\mathcal{H}$, equivalently $\gamma+\xi<N-2$ and the subcritical case corresponds to (δ,μ) under $\mathcal{H}$.

Remark 6.1 *The energy $\mathcal{E}_{H,0}$ of the particular solution associated to M_0 is always negative, given by $\mathcal{E}_{H,0}=-\frac{D}{(\mu+1)(\delta+1)}r^{N-2-\gamma-\xi}X_0Y_0(Z_0X_0)^{(\mu+1)/D}(W_0Y_0)^{(\delta+1)/D}$.*

Remark 6.2 *In the case $a = b = 0$, it is known that there exists a solution of the Dirichlet problem in any bounded regular domain Ω of $\mathbb{R}^N$, see for example [15], [20]; for general a, b, some restrictions on the coefficients appear, see [23] and [14].*

Next consider the critical and supercritical cases. When $a = b = 0$, there exists no solution if Ω is starshaped, see [36]. Here we show the existence of G.S. for general a, b. The existence in the critical case with $a = b = 0$ was first obtained in [22], then in the supercritical case in [29], and uniqueness was proved in [20], [29]. The proofs of [29] are quite long due to regularity problems, when δ or $\mu < 1$, which play no role in our quadratic system.

Remark 6.3 *The particular case $\delta = \mu$ and $a = b$ is easy to treat. Indeed in that case $u = v$ is a solution of the scalar equation $\Delta u + |x|^a |u|^{\delta-1} u = 0$, for which the critical case is given by $\delta = (N + 2 + 2a)/(N - 2)$. Moreover if system (SH) admits a G.S., or a solution of the Dirichlet problem in a ball, it satisfies $u = v$, from [3]. Then we are completely reduced to the scalar case. In particular, in the critical case, the G.S. are given explicitely by: $u = v = c(K + r^{(2+a)})^{(2-N)/(2+a)}$, where $K = c^{\delta-1}/(N + a)(N - 2)$; in other words they satisfy (3.3) with $X = Y$ and $Z = W$, i.e.*

$$\frac{X(t)}{N-2} + \frac{Z(t)}{N+a} - 1 = 0.$$

Near ∞, the G.S. is (obviously) symmetrical: it joins the points N_0 and A_0.

Remark 6.4 *Consider the case $\delta = 1$, $a = b = 0$, which is the case of the biharmonic equation*

$$\Delta^2 u = u^\mu.$$

Recall that it is the only case where the conjecture (1.3) was completely proved by Lin in [21]. In the critical case $\mu = (N + 4)/(N - 4)$, the G.S. are also given explicitely, see [20]:

$$u(r) = c(K + r^2)^{(4-N)/2}, \quad K = c^{\mu-1}/(N - 4)(N - 2)N(N + 2).$$

They satisfy the relation $XY = \frac{N-Z}{2}X + \frac{N-4}{2N}(N - W)Y$, and moreover we find that they are on an hyperplane, of equation

$$\frac{(N-2)X(t)}{N(N-4)} + \frac{Z(t)}{N} - 1 = 0.$$

Observe also that the G.S. is not symmetrical near ∞: u behaves like r^{4-N} and v behaves like r^{2-N}. The trajectory in the phase space joins the points N_0 and $Q_0 = (N - 4, N - 2, 2, 0)$.

Proof of Theorem 1.4. 1) *Existence or nonexistence results:*

• In the supercritical or critical case we apply any of the two conditions of Theorem 1.1: Here $\mathcal{E}_H(0) = 0$, and $\mathcal{E}_H$ is nonincreasing; there does not exist solutions of (M) such that at some time T, $X(T) = Y(T) = N - 2$, because at the time T,

$$XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} = (N-2)\left[N - 2 - \frac{N+a}{\delta+1} - \frac{N+b}{\mu+1} + \frac{W}{\mu+1} + \frac{Z}{\delta+1}\right] > 0$$

since $W > 0, Z > 0$, thus $\mathcal{E}_H(e^T) > 0$, which is impossible. Otherwise there exists no solution of the Dirichlet problem in a ball $B(0,R)$, because $\mathcal{E}_H(R) = R^N u'(R)v'(R) > 0$ from the Höpf Lemma. Then there exists a G.S. The uniqueness is proved in [20].

• In the subcritical case there is no radial G.S.: it would satisfy $\mathcal{E}_H(0) = 0$, and $\mathcal{E}_H$ is nondecreasing, $\mathcal{E}_H(r) \leqq Cr^{N-2-\gamma-\xi}$ from (5.1), and $\gamma + \xi > (N-2)$, then $\lim_{r\to\infty} \mathcal{E}_H(r) = 0$. From Theorem 1.1, there exists a solution of the Dirichlet problem.

2) *Behaviour of the G.S. in the critical case.*

It is easy to see that the condition (1.6) implies $\mu > \frac{2+b}{N-2}$ and $\delta > \frac{2+a}{N-2}$, and that $\delta \leqq \frac{N+a}{N-2}$ and $\mu \leqq \frac{N+b}{N-2}$ cannot hold simultaneously. One can suppose that $\delta > \frac{N+a}{N-2}$. Let $\mathcal{T}$ be the unique trajectory of the G.S.. Then $\mathcal{E}_H(0) = 0$, thus $\mathcal{T}$ lies on the variety $\mathcal{V}$ of energy 0, defined by

$$\frac{X(N+a-Z)}{\delta+1} + \frac{Y(N+b-W)}{\mu+1} = XY. \tag{6.2}$$

From (5.2) $\mathcal{T}$ starts from the point N_0, and from (5.1) $\mathcal{T}$ stays in

$$\mathcal{A} = \left\{(X,Y,Z,W) \in \mathbb{R}^4 : 0 < X < N-2, \quad 0 < Y < N-2, \quad 0 < Z < N+a, \quad 0 < W < N+b\right\}.$$

(i) Suppose that $\mathcal{T}$ converges to a fixed point of the system in $\bar{\mathcal{R}}$. Then the only possible points are A_0, P_0, Q_0 which are effectively on $\mathcal{V}$. Indeed I_0, $J_0, G_0, H_0 \notin \mathcal{V}$. But $Q_0 = ((N-2)\delta - (2+a), N-2, N+a-(N-2)\delta, 0) \notin \bar{\mathcal{R}}$, since $\delta > \frac{N+a}{N-2}$. And $P_0 \in \bar{\mathcal{R}}$ if and only if $\mu \leqq \frac{N+b}{N-2}$.

If $\mu > \frac{N+b}{N-2}$, then $\mathcal{T}$ converges to A_0. If $\mu < \frac{N+b}{N-2}$, no trajectory converges to A_0, from Proposition 4.5, thus $\mathcal{T}$ converges to P_0. If $\mu \neq \frac{N+b}{N-2}$ the convergence is exponential, thus the behaviour of u, v follows. If $\mu = \frac{N+b}{N-2}$, then $\mathcal{T}$ converges converges to $A_{0,} = P_0$; the eigenvalues given by (10.3) satisfy $\lambda_1 = \lambda_2 = N-2$, $\lambda_3 = N+a-\delta(N-2) < 0$ and $\lambda_4 = 0$; the projection of the trajectory on the hyperplane $Y = N-2$ satisfies the system

$$X_t = X\left[X - (N-2) + Z\right], \qquad Z_t = Z\left[N + a - \delta(N-2) - Z\right]$$

which presents a saddle point at $(N-2, 0)$, thus the convergence of X and Z is exponential, in particular we deduce the behaviour of u. The trajectory enters by the central variety of dimension 1, and by computation we deduce that $Y - (N-2) = -t^{-1} + O(t^{-2+\varepsilon})$ near ∞, and the behaviour of v follows.

(ii) Let us show that $\mathcal{T}$ converges to a fixed point. We eliminate W from (6.2) and we get a still quadratic system in (X,Y,Z) :

$$\left\{\begin{array}{c} X_t = X\left[X - (N-2) + Z\right], \\ Y_t = Y\left[Y + b + 2 - (\mu+1)X\right] + \frac{\mu+1}{\delta+1}X(N+a-Z), \\ Z_t = Z\left[N + a - \delta Y - Z\right]. \end{array}\right. \tag{6.3}$$

We have $X_t \geqq 0$, and $Y_t \geqq 0$ near $-\infty$. Suppose that X has a maximum at t_0 followed by a minimum at t_1. At these times $X_{tt} = XZ_t$, thus we find $Z_t(t_0) < 0 < Z_t(t_1)$. There exists $t_2 \in (t_0, t_1)$ such that $Z_t(t_2) = 0$, and t_2 is a minimum. At this time $Z(t_2) = N + a - \delta Y(t_2)$, $Z_{tt}(t_2) = -\delta(ZY_t)(t_2)$ hence

$$Y_t(t_2) = Y(t_2)\left[Y(t_2) + b + 2 - \frac{\mu+1}{\delta+1}X(t_2)\right] < 0$$

and $X_t(t_2) < 0$, hence $(X+Z)(t_2) < N-2$, and

$$N-2-X(t_2) > Z(t_2) > N+a-\delta(\frac{\mu+1}{\delta+1}X(t_2)-b-2)$$

$$(a+2)+\delta(b+2) < (\delta\frac{\mu+1}{\delta+1}-1)X(t_2) = \frac{\delta(2+b)+(2+a)}{(N-2)\delta-(2+a)}X(t_2)$$

but $X(t_2) < X(t_0) < \delta(N-2)-(2+a)$, which is contradictory. Then X has at most one extremum, which is a maximum, and then it has a limit in $(0, N-2]$ at ∞. In the same way, by symmetry, Y has at most one extremum, which is a maximum, and has a limit in $(0, N-2]$ at ∞. Then Z has at most one extremum, which is a minimum. Indeed at the points where $Z_t = 0$, $-Z_{tt}$ has the sign of Y_t. Thus Z has a limit in $[0, N+a)$, similarly W has a limit in $[0, N+b)$. ■

Open problems: 1) For the case $\delta = \mu$, in the critical case it is well known that there exist solutions (u,v) of system (SH) of the form (u,u), such that $r^\gamma u$ is periodic in $t = \ln r$. They correspond to a periodic trajectory for the scalar system (M_{scal}) with $p=2$, and it admits an infinity of such trajectories. If $\delta \neq \mu$, does there exist solutions (u,v) such that $(r^\gamma u, r^\xi v)$ is periodic in t, in other words a periodic trajectory for system (MH)?

2) In the supercritical case, we cannot prove that the regular trajectory $\mathcal{T}$ converges to M_0, that means $\lim_{r\to\infty} r^\gamma u = A$, $\lim_{r\to\infty} r^\xi v = B$. Here $\mathcal{E}_H(0) = 0$, $\mathcal{E}_H$ is nonincreasing, then $\mathcal{E}_H$ is negative. The only fixed points of negative energy are M_0, G_0, H_0, but a G.S. satisfies (5.5), then it tends to $(0,0)$ at ∞, hence $\mathcal{T}$ cannot converge to G_0 or H_0 from Proposition 4.9; but we cannot prove that $\mathcal{T}$ converges to some fixed point.

7 A nonvariational system

Here we consider system (S) with $p = q = 2, a = b$ and $s = m \neq 0$.

$$(SN)\begin{cases} -\Delta u = |x|^a\, u^s v^\delta, \\ -\Delta v = |x|^a\, u^\mu v^s, \end{cases}$$

where $D = \delta\mu - (1-s)^2 > 0$. In order to prove Theorem we can reduce the system to the case $a = 0$, by changing N into $\hat{N} = \frac{2(N+a)}{2+a}$, from Remark 2.4; thus we assume $a = 0$ in this Section. Here

$$X = -\frac{ru'}{u} \qquad Y = -\frac{rv'}{v}, \qquad Z(t) = -\frac{ru^s v^\delta}{u'}, \qquad W(t) = -\frac{ru^\mu v^s}{v'},$$

and system (M) becomes

$$(MN)\begin{cases} X_t = X\left[X-(N-2)+Z\right], \\ Y_t = Y\left[Y-(N-2)+W\right], \\ Z_t = Z\left[N-sX-\delta Y-Z\right], \\ W_t = W\left[N-\mu X-sY-W\right]. \end{cases}$$

We have chosen this system because it is not variational, and different hyperbolas in the plane (δ, μ):

• the hyperbola $\mathcal{H}_s$ for which the linearized system at M_0 has two imaginary roots, given by

$$(\mathcal{H}_s) \qquad \frac{1}{\delta+1-s}+\frac{1}{\mu+1-s}=\frac{N-2}{N-(N-2)s}$$

whenever $s<\frac{N}{N-2}$, and $\delta+1-s>0$, $\mu+1-s>0$, from Proposition 4.3;

• the hyperbola $\mathcal{H}_0$ defined by

$$(\mathcal{H}_0) \qquad \frac{1}{\delta+1}+\frac{1}{\mu+1}=\frac{N-2}{N}; \tag{7.1}$$

it was shown in [26] that above $\mathcal{H}_0$ there exists no solution of the Dirichlet problem;

• an hyperbola $\mathcal{Z}_s$ introduced in [38] in case $s<\frac{N}{N-2}$, and $\min(\delta,\mu)>|s-1|$:

$$(\mathcal{Z}_s) \qquad \frac{1}{\delta+1}+\frac{1}{\mu+1}=\frac{N-2}{N-(N-2)s}, \tag{7.2}$$

• we introduce the new curve $\mathcal{C}_s$ defined <u>*for any $s>0$*</u> by

$$(\mathcal{C}_s) \qquad \frac{N}{\mu+1}+\frac{N}{\delta+1}=N-2+\frac{(N-2)s}{2}\min(\frac{1}{\mu+1},\frac{1}{\delta+1}),$$

We first extend and complete the results of [38] and [26]:

Proposition 7.1 *(i) Assume $s<\frac{N}{N-2}$, and $\delta+1-s>0$, $\mu+1-s>0$. Under the hyperbola $\mathcal{Z}_s$, system (SN) admits no G.S., and then there is a solution of the Dirichlet problem in a ball.*

(ii) Above $\mathcal{H}_0$ there exists no solution of the Dirichlet problem. Thus there exists a G.S.

Proof. (i) We consider an energy function with parameters $\alpha,\beta,\sigma,\theta$:

$$\mathcal{E}_N(r)=r^N\left[u'v'+\alpha u^{\mu+1}v^s+\beta v^{\delta+1}u^s+\frac{\sigma}{r}vu'+\frac{\theta}{r}uv'\right] \tag{7.3}$$

$$=r^{N-2}uv\Psi_0=r^{N-2-\gamma-\xi}(ZX)^{\xi/2}(WZ)^{\gamma/2}\Psi_0, \tag{7.4}$$

from (4.2), where

$$\Psi_0(X,Y,Z,W)=XY+\alpha WY+\beta ZX-\sigma X-\theta Y. \tag{7.5}$$

We get

$$\begin{aligned} r^{1-N}(uv)^{-1}\mathcal{E}_N'(r)&=(\sigma+\theta-(N-2))XY+(N\alpha-\theta)YW+(N\beta-\sigma)XZ\\ &-(\alpha(\mu+1)-1)XYW-(\beta(\delta+1)-1)XYZ-\alpha sY^2W-\beta sX^2Z.\end{aligned}$$

Taking $\alpha=\frac{1}{\mu+1},\beta=\frac{1}{\delta+1}$, we find

$$r^{3-N}(uv)^{-1}\mathcal{E}_N'(r)=(\sigma+\theta-(N-2))XY+(N\alpha-\theta-\alpha sY)YW+(N\beta-\sigma-\beta sX)XZ. \tag{7.6}$$

If there exists a G.S., from (5.1) it satisfies $X,Y<N-2$, hence

$$r^{3-N}(uv)^{-1}\mathcal{E}_N'(r)>(\sigma+\theta-(N-2))XY+((N-(N-2)s)\alpha-\theta)YW+((N-(N-2)s)\beta-\sigma)XZ. \tag{7.7}$$

Taking $\theta = \frac{N-(N-2)s}{\mu+1}, \sigma = \frac{N-(N-2)s}{\delta+1}$, we deduce that $\mathcal{E}'_N > 0$ under $\mathcal{Z}_s$. Moreover $\mathcal{Z}_s$ is under $\mathcal{H}_s$, thus $\gamma + \xi > N-2$. Then $\mathcal{E}_N(r) = O(r^{N-2-\gamma-\xi})$ tends to 0 at ∞, which is contradictory.

(ii) Taking $\alpha = \frac{1}{\mu+1} = \frac{\theta}{N}, \beta = \frac{1}{\delta+1} = \frac{\sigma}{N}$, it comes from (7.6)

$$r^{3-N}(uv)^{-1}\mathcal{E}'_N(r) = (\frac{N}{\delta+1} + \frac{N}{\mu+1} - (N-2))XY - \alpha s Y^2 W - \beta s X^2 Z$$

hence $\mathcal{E}'_N < 0$ when (7.1) holds. At the value R where $u(R) = v(R) = 0$, we find $\mathcal{E}_N(R) = R^N u'(R)v'(R) > 0$, which is a contradiction. ■

Remark 7.2 *(i) When the four curves are simultaneously defined, they are in the following order, from below to above:* $\mathcal{Z}_s, \mathcal{H}_s, \mathcal{C}_s, \mathcal{H}_0$. *They intersect the diagonal* $\delta = \mu$ *repectively for*

$$\delta = \frac{N+2}{N-2} - 2s, \quad \delta = \frac{N+2}{N-2} - s, \quad \delta = \frac{N+2}{N-2} - \frac{s}{2}, \quad \delta = \frac{N+2}{N-2}.$$

(ii) For $\delta = \mu$, *system* (SN) *has a G.S. for* $\delta \geqq \frac{N+2}{N-2} - s$. *Indeed it admits solutions of the form* (U, U), *where* U *is a solution of equation* $-\Delta U = U^{s+\delta}$. *Suppose moreover* $s \leqq \delta$. *If* $1 - s < \delta < \frac{N+2}{N-2} - s$, *then there exists no G.S; indeed all such solutions satisfy* $u = v$, *from [3, Remark 3.3]. Then the point* $P_s = \left(\frac{N+2}{N-2} - s, \frac{N+2}{N-2} - s\right)$ *appears to be the separation point on the diagonal; notice that* $P_s \in \mathcal{H}_s$.

Next we prove our main existence result of existence of a G.S. valid without restrictions on s. The main idea is to introduce a new energy function Φ by adding two terms in X^2 and Y^2 to the energy $\mathcal{E}_N$ defined at (7.3). It is constructed in order that Φ' does not contain Y and Z. Then we consider the set of couples (X, Y) such that Φ' has a sign, which is bounded by a cubic curve. When (δ, μ) is above $\mathcal{C}_s$, the cubic curve is exterior to the square

$$K = [0, N-2] \times [0, N-2], \tag{7.8}$$

and then we can apply Theorem 1.1.

Proof of Theorem 1.5. From Theorem 1.1, if $s \geqq \frac{N+2}{N-2}$, all the regular solutions are G.S.. Thus we can assume $s < \frac{N+2}{N-2}$. Let $j, k \in \mathbb{R}$ be parameters, and

$$\begin{aligned}
\Phi(r) &= \mathcal{E}_N(r) + r^N \left[k\frac{s}{2}\frac{vu'^2}{u} + j\frac{s}{2}\frac{uv'^2}{v}\right] \\
&= r^N \left[u'v' + \alpha u^{\mu+1}v^s + \beta v^{\delta+1}u^s + \frac{\sigma}{r}vu' + \frac{\theta}{r}uv' + k\frac{s}{2}\frac{vu'^2}{u} + j\frac{s}{2}\frac{uv'^2}{v}\right] \\
&= r^{N-2}uv\Psi = r^{N-2-\gamma-\xi}(ZX)^{\xi/2}(WY)^{\gamma/2}\Psi,
\end{aligned}$$

where

$$\Psi(X, Y, Z, W) = XY + \alpha WY + \beta ZX - \sigma X - \theta Y + k\frac{s}{2}X^2 + j\frac{s}{2}Y^2.$$

Then

$$r^{3-N}(uv)^{-1}\Phi'(r) = (\sigma+\theta-(N-2))XY + (N\alpha-\theta)YW + (N\beta-\sigma)XZ$$
$$-(\alpha(\mu+1)-1)XYW - (\beta(\delta+1)-1)XYZ + (j-\alpha)sY^2W + (k-\beta)sX^2Z$$
$$+ksX^2\left[X-(N-2)\right] + jsY^2\left[Y-(N-2)\right] + (N-2-X-Y)(k\frac{s}{2}X^2 + j\frac{s}{2}Y^2).$$

We eliminate the terms in Z, W by taking $j = \alpha = \frac{1}{\mu+1}$, $k = \beta = \frac{1}{\delta+1}$, $\theta = N\alpha$, $\sigma = N\beta$. Then we get the function Φ defined at (1.9). Computing its derivative, we obtain after reduction

$$\mathcal{B}(X,Y) := -\frac{2}{s}r^{3-N}(uv)^{-1}\Phi'(r)$$
$$= \beta X^2(N-2-X) + \alpha Y^2(N-2-Y) + XY\left[\beta X + \alpha Y + \frac{2}{s}(N-2-N\alpha-N\beta)\right].$$

From Proposition 7.1 we can assume that $N(\alpha+\beta)-(N-2) > 0$. We determine the sign of $\mathcal{B}$ on the boundary ∂K of the square K defined at (7.8). We have $\mathcal{B}(0,Y) = \alpha Y^2(N-2-Y) \geqq 0$ and $\mathcal{B}(X,0) = \beta X^2(N-2-X) \geqq 0$. In particular $\mathcal{B}(0,0) = 0$. Otherwise $\mathcal{B}(N-2,Y) = Y\Theta(Y)$ with

$$\Theta(Y) = \alpha Y\left[2(N-2)-Y\right] + (N-2)((N-2)\beta + \frac{2}{s}(N-2-N\alpha-N\beta).$$

On the interval $[0, N-2]$, there holds $\Theta(Y) > \Theta(0)$. By hypothesis, (δ,μ) is above $\mathcal{C}_s$, or equivalently

$$(\alpha+\beta)\frac{N}{N-2} - 1 \leqq \frac{s}{2}\min(\alpha,\beta); \tag{7.9}$$

consequently $\mathcal{B}(N-2,Y) \geqq 0$ and similarly $\mathcal{B}(X,N-2) \geqq 0$. Then $\mathcal{B}$ is nonnegative on ∂K and is zero at $(0,0), (0,N-2), (N-2,0)$. The curve $\mathcal{B}(X,Y) = 0$ is a cubic with a double point at $(0,0)$, which is isolated under the condition (7.9): $\mathcal{B}(X,Y) > 0$ near $(0,0)$, except at this point. Then $\mathcal{B}(X,Y) > 0$ on the interior of K.

Suppose that there exists a regular solution such that $X(T) = Y(T) = N-2$ at the same time T. Indeed up to this time (X,Y) stays in K, thus the function Φ is decreasing. We have $\Phi(0) = 0$, and at the value $R = e^T$, we find

$$\Phi(R) = R^{N-2-\gamma-\xi}(N-2)^{\xi+\gamma+2}\left[\frac{\alpha W + \beta Z}{N-2} + 1 - (\beta+\alpha)(\frac{N}{N-2} - \frac{s}{2})\right]$$

then $\Phi(R) > 0$, since $\min(\alpha,\beta) < \alpha+\beta$. Therefore from Theorem 1.1, there exists a G.S. ∎

Remark 7.3 *We wonder if the limit curve for existence of G.S. would be $\mathcal{H}_s$, or another curve $\mathcal{L}_s$ defined by*

$$(\mathcal{L}_s) \qquad \frac{1}{\delta+1} + \frac{1}{\mu+1} = \frac{N-2}{N - \frac{(N-2)s}{2}},$$

which ensures that $\Phi(R) > 0$, and also $\mathcal{B}(N-2,N-2) > 0$. This curve cuts the diagonal at the same point $P_s = \left(\frac{N+2}{N-2} - s, \frac{N+2}{N-2} - s\right)$ as $\mathcal{H}_s$. Notice that $\mathcal{L}_s$ is under $\mathcal{H}_s$.

8 The radial potential system

Here we study the nonnegative radial solutions of system (SP) :

$$(SP) \begin{cases} -\Delta_p u = |x|^a u^s v^{m+1}, \\ -\Delta_q v = |x|^a u^{s+1} v^m, \end{cases}$$

with $a = b, \delta = m+1, \mu = s+1$, and we assume (1.5). System (M) becomes

$$(MP) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z \left[N + a - sX - (m+1)Y - Z \right], \\ W_t = W \left[N + b - (s+1)X - mY - W \right]. \end{cases}$$

For this system D, γ and ξ are defined by

$$D = p(1+m) + q(1+s) - pq, \qquad (p-1-s)\gamma + p + a = (m+1)\xi, \qquad (q-1-m)\xi + q + b = (s+1)\gamma,$$

thus γ and ξ are linked independtly of s, m by the relation

$$p(\gamma+1) = q(\xi+1) = \frac{pq(m+s+2+a)}{D}. \tag{8.1}$$

The system is *variational*. It admits an energy function, given at (1.13), which can also can be obtained by a direct computation in terms of X, Y, Z, W:

$$\mathcal{E}_P(r) = \psi \left[ZW - \frac{s+1}{p} W((N-p) - (p-1)X) - \frac{m+1}{q} Z((N-q) - (q-1)Y) \right], \tag{8.2}$$

where

$$\psi = \frac{r^{N-2-a} |u'|^{p-1} |v'|^{q-1}}{u^s v^m} = r^{N-(\gamma+1)p} \left[X^{q(s+1)(p-1)} Y^{p(m+1)(q-1)} Z^{p(q-m-1)} W^{q(p-s-1)} \right]^{1/D}.$$

Then we find

$$\mathcal{E}_P'(r) = (N + a - (s+1)\frac{N-p}{p} - (m+1)\frac{N-q}{q}) r^{N-1+a} u^{s+1} v^{m+1}.$$

Thus we define a critical line $\mathcal{D}$ as the set of $(\delta, \mu) = (m+1, s+1)$ such that

$$N + a = (m+1)\frac{N-q}{q} + (s+1)\,\frac{N-p}{p}, \tag{8.3}$$

equivalent to $pq(m+s+2+a) = ND$, or $N + a = (m+1)\xi + (s+1)\gamma$, or

$$(\gamma, \xi) = (\frac{N-p}{p}, \frac{N-q}{q})$$

The subcritical case is given by the set of points under $\mathcal{D}$, equivalently $\gamma > \frac{N-p}{p}$, $\xi > \frac{N-q}{q}$ or $(s+1)\,\gamma + (m+1)\xi > N + a$. The supercritical case is the set of points above $\mathcal{D}$.

Remark 8.1 *The energy* $(\mathcal{E}_{\mathcal{P}})_0$ *of the particular solution associated to* M_0 *is still negative:* $(\mathcal{E}_P)_0 = -\frac{D}{pq} r^{N+a-(\gamma+1)p} \left[X_0^{q(p-1)} Y_0^{p(q-1)} Z_0^{q(s+1)} W_0^{p(m+1)} \right]^{1/D}$.

Remark 8.2 *When* $p = q = 2$, *another energy function can be associated to the transformation given at Remark 2.2: the system (2.9) relative to* $u(r) = r^{-\gamma}\, U(t)$, $\quad v(r) = r^{-\xi}\, V(t)$ *is*

$$\begin{cases} \mathrm{U}_{tt} + (N-2-2\gamma)\mathrm{U}_t - \gamma(N-2-\gamma)\mathrm{U} + \mathrm{U}^s \mathrm{V}^{m+1} = 0 \\ \mathrm{V}_{tt} + (N-2-2\gamma)\mathrm{V}_t - \gamma(N-2-\gamma)\mathrm{V} + \mathrm{U}^{s+1} \mathrm{V}^{m} = 0 \end{cases} \tag{8.4}$$

and the function

$$E_P(t) = \frac{s+1}{2}(\mathrm{U}_t^2 - \gamma(N-2-\gamma)\mathrm{U}^2) + \frac{m+1}{2}(\mathrm{V}_t^2 - \gamma(N-2-\gamma)\mathrm{V}^2 + \mathrm{U}^{s+1\mathrm{V}m+1} \tag{8.5}$$

satisfies

$$(E_P)_t = -(N-2-2\gamma)\left[(s+1)\mathrm{U}_t^2 + (m+1)\mathrm{V}_t^2\right]$$

It differs from $\mathcal{E}_P$, *even in the critical case. This point is crucial for Section 9.*

It has been proved in [34], [35], that in the subcritical case with $a = 0$, there exists a solution of the Dirichlet problem in any bounded regular domain Ω of $\mathbb{R}^N$; and in the supercritical case there exists no solution if Ω is starshaped. Here we prove two results of existence or nonexistence of G.S. which seem to be new:

Proof of Theorem 1.6. 1) *Existence or nonexistence results.*

• In the supercritical or critical case there exists a G.S. From Theorem 1.1, if it were not true, then there would exist regular positive solutions of (MP) such that $X(T) = \frac{N-p}{p-1}$ and $Y(T) = \frac{N-q}{q-1}$. It would satisfy $\mathcal{E}_P \leqq 0$. Then at time T, we find $\mathcal{E}_P(R) > 0$, from (8.2), since $W > 0, Z > 0$, which is impossible.

• In the subcritical case, there exists no G.S. Suppose that there exists one. Now $\mathcal{E}_P$ is nondecreasing, hence $\mathcal{E}_P \geqq 0$. Its trajectory stays in the box $\mathcal{A}$ defined by (5.1), thus it is bounded. If $q \geqq m+1$ and $p \geqq s+1$, we deduce that , $\mathcal{E}_P(r) = O(r^{N-(\gamma+1)p})$ from (8.2), then $\mathcal{E}_P$ tends to 0 at ∞, which is contradictory. Next consider the general case. We have

$$\begin{aligned} \mathcal{E}_P(r) &\leqq r^{N-(\gamma+1)p} \left[X^{q(p-1)} Y^{p(q-1)} Z^{p(q-m-1)} W^{q(p-s-1)} \right]^{1/D} ZW \\ &= r^{N-(\gamma+1)p} \left[X^{q(p-1)} Y^{p(q-1)} Z^{q(1+s)} W^{p(1+m)} \right]^{1/D}, \end{aligned}$$

then the same result holds. Consequently, from Theorem 1.1, there exists a solution of the Dirichlet problem

2) *Behaviour of the G.S. in the critical case.*

Let $\mathcal{T}$ be the trajectory of a G.S.; then $\mathcal{E}_P(0) = 0$, thus $\mathcal{T}$ lies on the variety $\mathcal{V}$ of energy 0, also defined by

$$qW\left[(s+1)((p-1)X - (N-p)) + pZ\right] = p(m+1)Z\left[(N-q) - (q-1)Y)\right] \tag{8.6}$$

and $Y < \frac{N-q}{q-1}$, hence $(s+1)((p-1)X-(N-p))+pZ > 0$. From (5.2), $\mathcal{T}$ starts from $N_0 = (0,0,N+a,N+b)$ and stays in $\mathcal{A}$. Eliminating W in system (M), we find a system of three equations

$$\begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Y_t = YF, \\ Z_t = Z\left[N+a-sX-(m+1)Y-Z\right], \end{cases}$$

where

$$F(X,Y,Z) = \frac{1}{q}\left[\frac{N-q}{q-1} - Y\right]\frac{p(m+1-q)Z + q(s+1)((N-p)-(p-1)X)}{(s+1)((p-1)X-(N-p))+pZ}.$$

(i) If $\mathcal{T}$ converges to a fixed point of the system in $\bar{\mathcal{R}}$, the possible points on $\mathcal{V}$ are $A_0, I_0, J_0,$ $P_0,$ $Q_0, G_0, H_0,$ R_0, S_0. The eigenvalues of the linearized problem at A_0, given by (10.3) satisfy

$$\lambda_1,\lambda_2 > 0, \lambda_3 = N+a-s\frac{N-p}{p-1} - (m+1)\frac{N-q}{q-1} \leqq \lambda_4 = \lambda^* = N+a-(s+1)\frac{N-p}{p-1} - m\frac{N-q}{q-1},$$

since $q \leqq p$, and $\lambda_3 < \lambda^*$ for $q \neq p$, and $\lambda_3 = \lambda^* < 0$ for $q = p$, from (8.3). Then A_0 can be attained only when $\lambda^* \leqq 0$, from Proposition 4.5. And P_0 can be attained only if

$$q > m+1,\ \lambda^* \geqq 0 \ \text{ and } \ q+a < (s+1)\frac{N-p}{p-1}, \tag{8.7}$$

from Proposition 4.6, because $\gamma = \frac{N-p}{p} < \frac{N-p}{p-1}$. We observe that the condition $\lambda^* \geqq 0$ joint to (8.3) implies $m+1 < q < p$ and is equivalent to (8.7). Indeed it implies

$$\frac{N-p}{p-1}(s+1) \leqq N+a-m\frac{N-q}{q-1} = N+a-\frac{q}{q-1}(N+a-\frac{N-q}{q}-(s+1)\frac{N-p}{p});$$

then

$$(s+1)\,\frac{N-p}{p-1}\frac{q-p}{p} \leqq -(a+q),$$

thus $q < p$. From (8.3) we obtain

$$(N-q)(\frac{m+1}{q}-1) = q+a-(s+1)\frac{N-p}{p} \leqq (s+1)\,\frac{N-p}{p}(\frac{p-q}{p-1}-1) < 0,$$

hence $m+1 < q$ and (8.7) follows. By symmetry, Q_0 cannot be attained since $q \leqq p$. Then A_0 and P_0 are incompatible, unless $A_0 = P_0$, and P_0 is not attained when $p = q$.

(ii) Next we show that $\mathcal{T}$ converges to A_0 or to P_0. If t is an extremum value of Y, then

$$(\frac{m+1}{q}-1)Z(t) + \frac{s+1}{p}((N-p)-(p-1)X(t)) = 0. \tag{8.8}$$

This relation implies $q > m+1$ and

$$X_t(t) = \frac{X(t)Z(t)}{p-1}\left[1+\frac{p(m+1-q)}{q(s+1)}\right] = \frac{DX(t)Z(t)}{(p-1)q(s+1)} > 0.$$

In the same way, if t is an extremum value of X, then $p > s+1$ and $Y_t(t) > 0$. Near $-\infty$, there holds $X_t, Y_t \geqq 0$, and $Z_t, W_t \leqq 0$, from the linearization near N_0. Suppose that X has a maximum at t_0 followed by a minimum at t_1. Then $p > s+1$, and Y is increasing on $[t_0, t_1]$. At time t_0 we have $(p-1)X(t_0) + Z(t_0) = N - p$ and $X_{tt}(t_0) \leqq 0$, thus $Z_t(t_0) \leqq 0$; eliminating Z we deduce $p + a + (p-1-s)X(t_0) \leqq (m+1)Y(t_0)$ and similarly $(m+1)Y(t_1) \leqq p + a + (p-1-s)X(t_1)$; hence $Y(t_1) < Y(t_0)$, which is a contradiction. Thus X and Y can have at most one maximum, and in turn they have no maximum point. Therefore X and Y are increasing, and they are bounded, hence X has a limit in $\left(0, \frac{N-p}{p-1}\right]$ and Y has a limit in $\left(0, \frac{N-q}{q-1}\right]$. Then Z, W are decreasing; indeed at each time where $Z_t = 0$, we have $Z_{tt} = Z(-sX_t - (m+1)Y_t) < 0$, thus it is a maximum, which is impossible.

Then $\mathcal{T}$ converges to a fixed point of the system. Moreover, since X and Y are increasing, it cannot be one of the points $I_0, J_0, G_0, H_0, R_0, S_0$. It is necessarily A_0 or P_0. We distinguish two cases:

• Case $q \leqq m+1$. Then $\mathcal{T}$ converges to A_0, and $\lambda_3, \lambda^* < 0$, then (1.10) follows.

• Case $q > m+1$. Then $\mathcal{T}$ converges to A_0 (resp. P_0) when $\lambda^* \leqq 0$ (resp. $\lambda^* \geqq 0$). If the inequalities are strict, we deduce the convergence of u and v from Propositions 4.5 and 4.6, and (1.11) follows. If $\lambda^* = 0$, then $P_0 = A_0$, and $\lambda_3 = \frac{(N-1)(q-p)}{(p-1)(q-1)} < 0$. The projection of the trajectory $\mathcal{T}$ in $\mathbb{R}^3$ on the plane $Y = \frac{N-q}{q-1}$ satisfies the system

$$X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \qquad Z_t = Z\left[N + a - sX - (m+1)\frac{N-q}{q-1} - Z\right]$$

which presents a saddle point at $(\frac{N-p}{p-1}, 0)$, thus the convergence of X and Z is exponential, in particular we deduce the behaviour of u. The trajectory enters by the central variety of dimension 1, and by computation we deduce that $Y = \frac{N-q}{q-1} - \frac{1}{q-1-m}t^{-1} + O(t^{-2+\varepsilon})$, then (1.12) follows. ■

9 The nonradial potential system of Laplacians

Here we study the possibly <u>*nonradial*</u> solutions of the system of the preceeding Section when $p = q = 2$:

$$(SL)\begin{cases} -\Delta u = |x|^a \, u^s v^{m+1}, \\ -\Delta v = |x|^a \, u^{s+1} v^m, \end{cases}$$

with $D = s + m$. We solve an open problem of [7]: the nonexistence of (radial or nonradial) G.S. under condition (1.14).

It was shown in [7] in the case $N + a \geqq 4$. The problem was open when $N + a < 4$, and $m+s+1 > (N+a)/(N-2)$, which implies $N < 6$. Indeed in the case $m+s+1 \leqq (N+a)/(N-2)$, there are no solutions of the exterior problem, see [6, Theorem 5.3]. Recall that the main result of [7] is the obtention of apriori estimates near 0 or ∞, by using the Bernstein technique introduced in [18] and improved in [8]. Then the behaviour of the solutions is obtained by using the change of unknown

$$u(r,\theta) = r^{-\gamma}\mathrm{U}(t,\theta), \qquad v(r,\theta) = r^{-\gamma}\mathrm{V}(t,\theta), \qquad t = \ln r,$$

extending the transformation of Remark 8.2 to the nonradial case (in fact here t is $-t$ in [7]); it leads to the system

$$\mathrm{U}_{tt}+(N-2-2\gamma)\mathrm{U}_t+\Delta_S\mathrm{U}-\gamma(N-2-\gamma)\mathrm{U}+\mathrm{U}^s\mathrm{V}^{m+1}=0,$$
$$\mathrm{V}_{tt}+(N-2-2\gamma)\mathrm{V}_t+\Delta_S\mathrm{V}-\gamma(N-2-\gamma)\mathrm{V}+\mathrm{U}^{s+1}\mathrm{V}^{m}=0,$$

where Δ_S is the Laplace-Beltrami operator on S_{N-1}. A corresponding energy is introduced in [7]:

$$\begin{aligned}E_L(t)&=\frac{s+1}{2}\int\limits_{S^{N-1}}(\mathrm{U}_t^2-|\nabla_S\mathrm{U}|^2-\gamma(N-2-\gamma)\mathrm{U}^2)d\theta\\&+\frac{m+1}{2}\int\limits_{S^{N-1}}(\mathrm{V}_t^2-|\nabla_S\mathrm{V}|^2-\xi(N-2-\xi)\mathrm{V}^2)d\theta+\int\limits_{S^{N-1}}\mathrm{U}^{s+1}\mathrm{V}^{m+1}d\theta,\end{aligned}$$

extending (8.5) to the nonradial case; it satisfies

$$(E_L)_t=-(N-2-2\gamma)\int\limits_{S^{N-1}}\left[(s+1)\mathrm{U}_t^2+(m+1)\mathrm{V}_t^2\right]d\theta$$

Here we construct <u>*another energy function*</u>, extending the Pohozaev function defined at (1.13) to the nonradial case.

Lemma 9.1 *Consider the function $\mathcal{E}_L(r)$ defined by*

$$\begin{aligned}r^{-N}\mathcal{E}_L(r)&=\frac{s+1}{2}\int\limits_{S^{N-1}}\left[u_r^2-r^{-2}\left|\nabla_S u\right|^2+(N-2)\frac{uu_r}{r}\right]d\theta\\&+\frac{m+1}{2}\int\limits_{S^{N-1}}\left[(\frac{\partial v}{\partial\nu})^2-r^{-2}\left|\nabla_S v\right|^2+(N-2)\frac{vv_r}{r}\right]d\theta+r^a\int\limits_{S^{N-1}}u^{s+1}v^{m+1}d\theta.\end{aligned}$$

Then the following relation holds:

$$r^{1-N}\mathcal{E}_L'(r)=(N+a-(s+1)\frac{N-2}{2}-(m+1)\frac{N-2}{2})r^a\int\limits_{S^{N-1}}u^{s+1}v^{m+1}d\theta.$$

Proof. In terms of t, we find

$$\begin{aligned}&\mathcal{E}_L(t)=\mathcal{E}_{L,1}(t)+\mathcal{E}_{L,2}(t)+\mathcal{E}_{L,3}(t),\ \text{with}\\&\mathcal{E}_{L,1}(t)=\frac{s+1}{2}e^{(N-2)t}\int\limits_{S^{N-1}}\left[u_t^2-|\nabla_S u|^2+(N-2)uu_t\right]d\theta,\\&\mathcal{E}_{L,2}(t)=\frac{m+1}{2}e^{(N-2)t}\int\limits_{S^{N-1}}\left[v_t^2-|\nabla_S v|^2+(N-2)vv_t\right]d\theta,\quad\mathcal{E}_{L,3}(t)=e^{(N+a)t}\int\limits_{S^{N-1}}u^{s+1}v^{m+1}d\theta,\end{aligned}$$

and u satisfies the equations

$$u_{tt}+(N-2)u_t+\Delta_S u+e^{(2+a)t}u^sv^{m+1}=0,\tag{9.1}$$
$$(e^{(N-2)t}u_t)_t+e^{(N-2)t}\Delta_S u+e^{(N+a)t}u^sv^{m+1}=0,\tag{9.2}$$

and v satisfies symmetrical equations. Multiplying (9.2) by u and (9.1) by $(s+1)e^{(N-2)t}u_t$, we obtain

$$\begin{aligned}0 &= \int_{S^{N-1}} u(e^{(N-2)t}u_t)_t + e^{(N-2)t}\int_{S^{N-1}} u\Delta_S u + e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1} \\ &= \frac{d}{dt}\int_{S^{N-1}} ue^{(N-2)t}u_t - e^{(N-2)t}\int_{S^{N-1}} (u_t^2 + |\nabla_S u|^2) + e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1}\end{aligned}$$

$$\begin{aligned}&\frac{d}{dt}\int_{S^{N-1}} \frac{s+1}{2}(N-2)ue^{(N-2)t}u_t - \frac{s+1}{2}(N-2)e^{(N-2)t}\int_{S^{N-1}} (u_t^2 + |\nabla_S u|^2) \\ &= -\frac{s+1}{2}(N-2)e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1},\end{aligned}$$

and symmetrically for v, and adding the equalities we deduce

$$\begin{aligned}0 &= (s+1)(e^{(N-2)t}\frac{d}{dt}\int_{S^{N-1}} (\frac{u_t^2 - |\nabla_S u|^2}{2} + (N-2)e^{(N-2)t}\int_{S^{N-1}} u_t^2 \\ &+ (m+1)(e^{(N-2)t}\frac{d}{dt}\int_{S^{N-1}} \frac{v_t^2 - |\nabla_S v|^2}{2} + (N-2)e^{(N-2)t}\int_{S^{N-1}} v_t^2 \\ &+ \frac{d}{dt}(e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1}) - (N+a)e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1}\end{aligned}$$

$$\begin{aligned}&\frac{d}{dt}\left[\frac{e^{(N-2)t}}{2}\int_{S^{N-1}} ((s+1)(u_t^2 - |\nabla_S u|^2) + (m+1)(v_t^2 - |\nabla_S v|^2)) + e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1}\right] \\ &+ \frac{N-2}{2}e^{(N-2)t}\int_{S^{N-1}} ((s+1)(u_t^2 + |\nabla_S u|^2) + (m+1)(v_t^2 + |\nabla_S v|^2)) \\ &= (N+a)e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1},\end{aligned}$$

hence

$$(\mathcal{E}_L)_t(t) = (N + a - (s+1)\frac{N-2}{2} - (m+1)\frac{N-2}{2})e^{(N+a)t}\int_{S^{N-1}} u^{s+1}v^{m+1}d\theta.$$

■

Proof of Theorem 1.7. Suppose that there exists a G.S. Since $s+m+1 < (N+2+2a)/(N-2)$ we deduce that E_L and $\mathcal{E}_L$ are increasing and start from 0, then they stay positive. From [7, Corollary 6.4], since $s+m+1 < (N+2)/(N-2)$, three eventualities can hold. The first one is that (u,v) behaves like the particular solution (u_0, v_0); it cannot hold because E_L has a negative limit, see [7, Remark 6.3]. The second one is that (u,v) is regular at ∞, that means $\lim_{|x|\to\infty} |x|^{N-2} u = \alpha > 0$, $\lim_{|x|\to\infty} |x|^{N-2} v = \beta > 0$; it cannot hold because $\lim_{t\to\infty} E_L(t) = 0$. It remains a third eventuality: when for example $m > (N+a)/(N-2)$, and (u,v) has the following behaviour at ∞ :

$$\lim_{r\to\infty} u = \alpha > 0, \text{ and } \lim_{|x|\to\infty} |x|^k v = \beta > 0 \text{ or } 0, \qquad \text{with } k = (2+a)/(m-1). \tag{9.3}$$

The condition on m implies that $N < 4-a$ from assumption (1.14). In that case $\lim_{t\to\infty} E_L(t) = \infty$, which gives no contradiction. Here we show that a contradiction holds by using the new energy function $\mathcal{E}_L$.

First recall the proof of (9.3). Making the substitution

$$u(r,\theta) = u(t,\theta), \qquad v(r,\theta) = r^{-k}\mathbf{V}(t,\theta), \qquad t = \ln r, \theta \in S_{N-1},$$

we get

$$\left\{ \begin{array}{c} u_{tt} + (N-2)u_t + \Delta_S u + e^{-2kt} u^s \mathbf{V}^{m+1} = 0, \\ \mathbf{V}_{tt} + (N-2-2k)\mathbf{V}_t + \Delta_S \mathbf{V} - k(N-2-k)\mathbf{V} + u^{s+1}\mathbf{V}^m = 0. \end{array} \right. \tag{9.4}$$

Then $u, \mathbf{V}$ are bounded near ∞, and from [7, Proposition 4.1] u converges exponentially to the constant α, more precisely

$$\| |u-\alpha| + |u_t| + |\nabla_S u| \|_{C^0(S^{N-1})} = O(e^{-(N-2)t}), \tag{9.5}$$

because $k \neq (N-2)/2$ and all the derivatives of $\mathbf{V}$ up to the order 2 are bounded. The equation in $\mathbf{V}$ takes the form

$$\mathbf{V}_{tt} + (N-2-2k)\mathbf{V}_t + \Delta_S \mathbf{V} - k(N-2-k)\mathbf{V} + \alpha^{s+1}\mathbf{V}^m + \varphi = 0$$

where φ and its derivatives up to the order 2 are $O(e^{-(N-2)t})$. From [7, Theorem 4.1], the function $\mathbf{V}$ converges to β or to 0 in $C^2(S^{N-1})$.

Next we define

$$f(t) = e^{(N-2)t} \int_{S^{N-1}} u_t d\theta = r^{N-1} \int_{S^{N-1}} u_r d\theta.$$

Then

$$\mathcal{E}_{L,1}(t) = (N-2)\frac{s+1}{2}\alpha f(t) + O((e^{-(N-2)t}))$$

from (9.5). Moreover from (9.4),

$$f_t(t) = -e^{(N-2-2k)t} \int_{S^{N-1}} u^s \mathbf{V}^{m+1} d\theta < 0.$$

Since u is regular at 0, $f(t) = 0(e^{(N-1)t})$ at $-\infty$, in particular $\lim_{t\to-\infty} f(t) = 0$. And $f_t(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$ at ∞, then $f(t)$ has a finite negative limit $-\ell^2$; and

$$\lim_{t\to\infty} \mathcal{E}_{L,1}(t) = -(N-2)\frac{s+1}{2}\alpha\ell^2.$$

Moreover $v = e^{-kt}\mathbf{V}$, and $\mathbf{V}$ and its derivatives up to the order 2 are bounded, thus

$$\mathcal{E}_{L,2}(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$$

Finally

$$\mathcal{E}_{L,3}(t) = O(e^{(N+a-k(m+1))t})$$

and $N+a-k(m+1) < \frac{2-N}{m-1} < 0$. Then $\mathcal{E}_L$ has a finite limit $\theta < 0$ at ∞, which is contradictory. ■

10 Analysis of the fixed points

Here we make the local analysis around the fixed points.

Proof of Proposition 4.4. (i) Consider a regular solution (u,v) with initial data (u_0,v_0). When when $r\to 0$, we have

$$(-r^{N-1}\left|u'\right|^{p-2}u')'=r^{N-1+a}u_0^s v_0^\delta(1+o(1)),\qquad -\left|u'\right|^{p-2}u'=\frac{1}{N+a}r^{1+a}u_0^s v_0^\delta(1+o(1)),$$

thus from (2.1), when $t\to-\infty$

$$X(t)=(\frac{1}{N+a}u_0^{s+1-p}v_0^\delta)^{1/(p-1)}e^{(p+a)t/(p-1)}(1+o(1)),$$
$$Y(t)=(\frac{1}{N+b}u_0^{\mu}v_0^{m+1-q})^{1/(q-1)}e^{(q+b)t/(q-1)}(1+o(1)),$$

and $\lim_{t\to-\infty}Z=N+a$, $\lim_{t\to-\infty}W=(N+b)$. In particular the trajectory tends to $N_0=(0,0,N+a,N+b)$.

(ii) Reciprocally, consider a trajectory converging to N_0. Setting $Z=N+a+\tilde{Z},W=N+b+\tilde{W}$, the linearized system is

$$X_t=\frac{p+a}{p-1}X,\quad Y_t=\frac{q+b}{q-1}Y,\quad \tilde{Z}_t=(N{+}a)\left[-sX-\delta Y-\tilde{Z}\right],\quad \tilde{W}_t=(N{+}b)\left[-\mu X-mY-\tilde{W}\right]. \tag{10.1}$$

The eigenvalues are

$$\lambda_1=\frac{p+a}{p-1}>0,\quad \lambda_2=\frac{q+b}{q-1}>0,\quad \lambda_3=-(N+a)<0,\quad \lambda_4=-(N+b)<0. \tag{10.2}$$

The unstable variety $\mathcal{V}_u$ and the stable variety $\mathcal{V}_s$ have dimension 2. Notice that $\mathcal{V}_s$ is contained in the set $X=Y=0$, thus no admissible trajectory converges to N_0 when $r\to\infty$, and there exists an infinity of admissible trajectories in $\mathcal{R}$, converging to N_0 when $r\to 0$. Moreover we get $\lim_{t\to-\infty}e^{-(p+a)/(p-1)t}X(t)=\kappa>0$ and $\lim_{t\to-\infty}e^{-(q+b)/(q-1)t}Y(t)=\ell>0$. Thus (u,v) have a positive limit $(u_0,v_0)=((N+a)\kappa^{p-1})^{(q-1-m)/D}((N+b)\ell^{q-1})^{\delta/D}$ from (4.2), (4.1), hence (u,v) is a regular solution.

Next we show that *for any* $\kappa>0,\ell>0$ there exists a unique local solution such that $\lim_{t\to-\infty}e^{-(p+a)t/(p-1)}X(t)=\kappa$ and $\lim_{t\to-\infty}e^{-(q+b)/(q-1)t}Y=\ell$. On $\mathcal{V}_u$, we get a system of two equations of the form

$$X_t=X(\lambda_1+F(X,Y)),\quad Y_t=Y(\lambda_2+G(X,Y)),$$

where $F=AX+BY+f(X,Y)$, where f is a smooth function with $f_X(0,0)=f_Y(0,0)=0$, similarly for G. Setting $X=e^{\lambda_1 t}(\kappa+x)$, $Y=e^{\lambda_2 t}(\ell+y)$, and assuming $\lambda_2\geqq\lambda_1$ and setting $\rho=e^{\lambda_1 t}$ we obtain

$$x_\rho=\frac{1}{\rho}(\kappa+x)F(\rho(\kappa+x),\rho^{\lambda_2/\lambda_1}(\ell+y)),\qquad y_\rho=(\ell+y)G(\rho(\kappa+x),\rho^{\lambda_2/\lambda_1}(\ell+y)),$$

with $x(0) = y(0) = 0$. Then we get local existence and uniqueness. Hence for any $u_0, v_0 > 0$ there exists a regular solution (u, v) with initial data (u_0, v_0). Moreover $u, v \in C^1([0, R))$ when $a, b > -1$.

■

Proof of Proposition 4.5. The linearization at $A_0 = \left(\frac{N-p}{p-1}, \frac{N-q}{q-1}, 0, 0\right)$ gives, with $X = \frac{N-p}{p-1} + \tilde{X}, Y = \frac{N-q}{q-1} + \tilde{Y}$,

$$\tilde{X}_t = \frac{N-p}{p-1}\left[\tilde{X} + \frac{Z}{p-1}\right], \quad \tilde{Y}_t = \frac{N-q}{q-1}\left[\tilde{Y} + \frac{W}{q-1}\right], \quad Z_t = \lambda_3 Z, \quad W_t = \lambda_4 W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \ \lambda_2 = \frac{N-q}{q-1} > 0, \ \lambda_3 = N+a-s\frac{N-p}{p-1}-\delta\frac{N-q}{q-1}, \ \lambda_4 = N+b-\mu\frac{N-p}{p-1}-m\frac{N-q}{q-1}. \tag{10.3}$$

• Convergence when $r \to \infty$: If $\lambda_3 > 0$, or $\lambda_4 > 0$, then the stable variety$\mathcal{V}_s$ has at most dimension 1, it satisfies $W = 0$ or $Z = 0$, hence there is no admissible trajectory converging to A_0 at ∞. If $\lambda_3 < 0$, and $\lambda_4 < 0$, then $\mathcal{V}_s$ has dimension 2. Moreover $\mathcal{V}_s \cap \{Z = 0\}$ has dimension 1 : the corresponding system in X, Y, W has the eigenvalues $\lambda_1, \lambda_2, \lambda_4$; similarly $\mathcal{V}_s \cap \{W = 0\}$ has dimension 1. Then there exist trajectories in $\mathcal{V}_s$ such that $Z > 0$ and $W > 0$, included in $\mathcal{R}$ and thus admissible. They satisfy $\lim e^{-\lambda_3 t} Z = C_3 > 0, \lim e^{-\lambda_4 t} W = C_4 > 0$, then (4.11) follows from (4.2).

• Convergence when $r \to 0$: If $\lambda_3 < 0$, or $\lambda_4 < 0$, the unstable variety $\mathcal{V}_u$ has at most dimension 3, and it satisfies $W = 0$ or $Z = 0$. Therefore there is no admissible trajectory converging at $-\infty$. If $\lambda_3, \lambda_4 > 0$, then $\mathcal{V}_u$ has dimension 4; in that case there exist admissible trajectories, and (4.11) follows as above. ■

Proof of Proposition 4.6. We set $P_0 = \left(\frac{N-p}{p-1}, Y_*, 0, W_*\right)$, with

$$Y_* = \frac{\frac{N-p}{p-1}\mu - (q+b)}{q-1-m}, \qquad W_* = \frac{(q-1)(N+b-\frac{N-p}{p-1}\mu) - m(N-q)}{q-1-m},$$

for $m + 1 \neq q$. The linearization at P_0 gives, with $X = \frac{N-p}{p-1} + \tilde{X}, Y = Y_* + \tilde{Y}$, $W = W_* + \tilde{W}$,

$$\tilde{X}_t = \frac{N-p}{p-1}\left[\tilde{X} + \frac{Z}{p-1}\right], \quad \tilde{Y}_t = Y_*\left[\tilde{Y} + \frac{\tilde{W}}{q-1}\right], \quad Z_t = \lambda_3 Z, \quad \tilde{W}_t = W_*\left[-\mu\tilde{X} - m\tilde{Y} - \tilde{W}\right]$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_3 = N + a - s\frac{N-p}{p-1} - \delta Y_* = \frac{D}{q-1-m}(\gamma - \frac{N-p}{p-1}),$$

and the roots λ_2, λ_4 of equation

$$\lambda^2 - (Y_* - W_*)\lambda + \frac{m+1-q}{q-1}Y_* W_* = 0$$

Then if $\lambda_3 < 0$ (resp. $\lambda_3 > 0$) there is no admissible trajectory converging when $r \to 0$ (resp. $r \to \infty$). Indeed $\mathcal{V}_u = \mathcal{V}_u \cap \{Z = 0\}$ (resp. $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$).

1) Suppose that $q > m+1$. Since $q+b < \frac{N-p}{p-1}\mu < N+b-m\frac{N-q}{q-1}$, we have $P_0 \in \mathcal{R}$, and $\lambda_2\lambda_4 < 0$. First assume $\lambda_3 < 0$, that means $\gamma < \frac{N-p}{p-1}$. Then $\mathcal{V}_s$ has dimension 2, and $\mathcal{V}_s \cap \{Z=0\}$ has dimension 1, there exists trajectories with $Z > 0$, which are admissible, converging when $r \to \infty$. Next assume $\lambda_3 > 0$. Then $\mathcal{V}_u$ has dimension 3, and $\mathcal{V}_u \cap \{Z=0\}$ has dimension 2. Thus there exist admissible trajectories converging when $t \to -\infty$.

2) Suppose that $q < m+1$. Since $q+b > \frac{N-p}{p-1}\mu > N+b-m\frac{N-q}{q-1}$, we have $P_0 \in \mathcal{R}$, and $\lambda_2\lambda_4 > 0$. We assume $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1)+(b+1)q$, that means $Y_* \neq W_*$. First suppose $\lambda_3 > 0$, that means $\gamma < \frac{N-p}{p-1}$. If $\mathrm{Re}\lambda_2 > 0$, then $\mathcal{V}_u$ has dimension 4, or $\mathrm{Re}\lambda_2 < 0$ then $\mathcal{V}_u$ has dimension 2 and $\mathcal{V}_u \cap \{Z=0\}$ has dimension 1. In any case, there exist admissible trajectories converging when $r \to 0$. Next assume $\lambda_3 < 0$. If $\mathrm{Re}\lambda_2 > 0$, then $\mathcal{V}_s$ has dimension 1, and $\mathcal{V}_s \cap \{Z=0\} = \emptyset$. If $\mathrm{Re}\lambda_2 < 0$, then $\mathcal{V}_s$ has dimension 3. In any case $\mathcal{V}_s$ contains trajectories with $Z > 0$, which are admissible, converging when $r \to \infty$.

Those trajectories satisfy $\lim e^{-\lambda_3 t}Z = C_3 > 0$, $\lim X = \frac{N-p}{p-1}$, $\lim Y = Y_*$ and $\lim W = W_*$, thus (4.12) follows from (4.2) and (2.5). ∎

Proof of Proposition 4.8. The linearization at $I_0 = (\frac{N-p}{p-1},0,0,0)$ gives, with $X = \frac{N-p}{p-1} + \tilde{X}$,

$$\tilde{X}_t = \frac{N-p}{p-1}(\tilde{X}+\frac{Z}{p-1}), \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a-s\frac{N-p}{p-1})Z, \quad W_t = (N+b-\mu\frac{N-p}{p-1})W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = -\frac{N-q}{q-1} < 0, \quad \lambda_3 = N+a-s\frac{N-p}{p-1}, \quad \lambda_4 = N+b-\mu\frac{N-p}{p-1}.$$

• Convergence when $r \to \infty$: If $\lambda_3 > 0$ or $\lambda_4 > 0$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{Z=0\}$ or $\mathcal{V}_s = \mathcal{V}_s \cap \{W=0\}$. There is no admissible trajectory converging at ∞. Next suppose that $\lambda_3, \lambda_4 < 0$. Then $\mathcal{V}_s$ has dimension 3; it contains trajectories with $Y, Z, W > 0$, which are admissible. They satisfy $\lim X = \frac{N-p}{p-1}$, $\lim e^{-\lambda_2 t}Y = C_2 > 0$, $\lim e^{-\lambda_3 t}Z = C_3 > 0$, $\lim e^{-\lambda_4 t}W = C_4 > 0$, then (4.13) follows from (4.2) and (2.4).

• Convergence when $r \to 0$: Since $\lambda_2 < 0$ we have $\mathcal{V}_u = \mathcal{V}_u \cap \{Y=0\}$, hence there is no admissible trajectory converging when $r \to 0$. ∎

Proof of Proposition 4.9. The point $G_0 = (\frac{N-p}{p-1},0,0,N+b-\frac{N-p}{p-1}\mu) \in \mathcal{R}$ since $\frac{N-p}{p-1}\mu < N+b$. The linearization at G_0 gives, with $X = \frac{N-p}{p-1} + \tilde{X}, W = N+b-\frac{N-p}{p-1}\mu + \tilde{W}$,

$$\begin{aligned}\tilde{X}_t &= \frac{N-p}{p-1}\left[\tilde{X} + \frac{Z}{p-1}\right], \quad Y_t = \frac{Y}{q-1}(q+b-\frac{N-p}{p-1}\mu), \\ Z_t &= (N+a-s\frac{N-p}{p-1})Z, \quad W_t = (N+b-\frac{N-p}{p-1}\mu)\left[-\mu\tilde{X} - mY - \tilde{W}\right]\end{aligned}$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \ \lambda_2 = \frac{1}{q-1}(q+b-\frac{N-p}{p-1}\mu), \ \lambda_3 = N+a-s\frac{N-p}{p-1}, \ \lambda_4 = \frac{N-p}{p-1}\mu - N - b < 0.$$

• Convergence when $r \to \infty$: If $\lambda_2 > 0$, or $\lambda_3 > 0$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{Y=0\}$ or $\mathcal{V}_s = \mathcal{V}_s \cap \{Z=0\}$, there is no admissible trajectory converging at ∞. Assume $\lambda_2, \lambda_3 < 0$, then $\mathcal{V}_s$ has dimension 3, it contains trajectories with $Y, Z > 0$, which are admissible.

• Convergence when $r \to 0$: If $\lambda_3 < 0$, or $\lambda_2 < 0$ there is no admissible trajectory. If $\lambda_2, \lambda_3 > 0$ then $\mathcal{V}_s$ has dimension 3, it contains admissible trajectories.

In any case $\lim X = \frac{N-p}{p-1}$, $\lim e^{-\lambda_2 t} Y = C_2 > 0$, $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim W = N + b - \frac{N-p}{p-1}\mu$, hence (4.13) still follows from (4.2) and (2.4). ∎

Proof of Proposition 4.10. We set $C_0 = \left(0, \bar{Y}, 0, \bar{W}\right)$, with

$$\bar{Y} = \frac{q+b}{m+1-q}, \qquad \bar{W} = \frac{m(N-q)-(N+b)(q-1)}{m+1-q}. \tag{10.4}$$

Then $C_0 \in \mathcal{R}$ if $\frac{N-q}{q-1}m > N+b$, implying $q < m+1$. The linearization at C_0 gives, with $Y = \bar{Y} + \tilde{Y}$ and $W = \bar{W} + \tilde{W}$

$$X_t = -\frac{N-p}{p-1}X, \quad \tilde{Y}_t = \bar{Y}\left[\tilde{Y} + \frac{\tilde{W}}{q-1}\right], \quad Z_t = \lambda_3 Z, \quad W_t = \bar{W}\left[-\mu X - m\tilde{Y} - \tilde{W}\right].$$

The eigenvalues are

$$\lambda_1 = -\frac{N-p}{p-1}, \quad \lambda_3 = N + a - \delta\bar{Y},$$

and the roots λ_2, λ_4 of equation

$$\lambda^2 - (\bar{Y} - \bar{W})\lambda + \frac{m+1-q}{q-1}\bar{Y}\bar{W} = 0 \tag{10.5}$$

then $\lambda_2\lambda_4 > 0$. We assume $m \neq \frac{N(q-1)+(b+1)q}{N-q}$, that means $\bar{Y} \neq \bar{W}$.

• Convergence when $r \to \infty$: if $\lambda_3 > 0$ we have $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$, hence there is no admissible trajectory. Next assume that $\lambda_3 < 0$, that means $\delta > (N+a)\frac{m+1-q}{q+b}$.If $\operatorname{Re}\lambda_2 < 0$ (resp. > 0) then $\mathcal{V}_s$ has dimension 4 (resp. 2) and $\mathcal{V}_s \cap \{X = 0\}$ and $\mathcal{V}_s \cap \{Z = 0\}$ have dimension 3 (resp. 1) then there exist trajectories with $X, Z > 0$, which are admissible.

In any case $\lim e^{-\lambda_1 t} X = C_1 > 0$, $\lim Y = \bar{Y}$, $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim W = \bar{W}$, then (4.15) follows.

• Convergence when $r \to 0$: Since $\lambda_1 < 0$ we have $\mathcal{V}_u = \mathcal{V}_u \cap \{X = 0\}$, hence there is no admissible trajectory. ∎

Proof of Proposition 4.11. We set $R_0 = \left(0, \bar{Y}, \bar{Z}, \bar{W}\right)$, where $\bar{Y}, \bar{W}$ are defined at (10.4), and $\bar{Z} = N + a - \delta\frac{b+q}{m+1-q}$.Under our assumptions it lies in $\mathcal{R}$. Setting $Y = \bar{Y} + \tilde{Y}, Z = \bar{Z} + \tilde{Z}, W = \bar{W} + \tilde{W}$, the linearization at R_0 gives

$$X_t = \lambda_1 X, \quad \tilde{Y}_t = \bar{Y}\left[\tilde{Y} + \frac{\tilde{W}}{q-1}\right], \quad Z_t = \bar{Z}\left[-sX - \delta\tilde{Y} - \tilde{Z}\right], \quad W_t = \bar{W}\left[-\mu X - m\tilde{Y} - \tilde{W}\right];$$

the eigenvalues are

$$\lambda_1 = \frac{1}{p-1}(p + a - \delta\frac{b+q}{m+1-q}), \quad \lambda_3 = -\bar{Z} < 0;$$

and the roots λ_2, λ_4 of equation of equation (10.5).

• Convergence when $r \to \infty$: If $\lambda_1 > 0$, that means $(p+a)\frac{m+1-q}{q+b} < \delta$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{X=0\}$, hence there is no admissible trajectory. Next assume $\lambda_1 < 0$; if $\operatorname{Re}\lambda_2 < 0$ (resp. > 0) then $\mathcal{V}_s$ has dimension 4(resp. 2) and $\mathcal{V}_s \cap \{X=0\}$ has dimension 3 (resp. 1) then there exist admissible trajectories.

• Convergence when $r \to 0$: If $\lambda_1 < 0$, then $\mathcal{V}_u = \mathcal{V}_u \cap \{X=0\}$, hence there is no admissible trajectory. Next assume $\lambda_1 > 0$. If $\operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_4 < 0$ (resp. > 0) then $\mathcal{V}_s$ has dimension 4 (resp. 2) and $\mathcal{V}_s \cap \{X=0\}$ has dimension 3 (resp. 1) then there exist admissible trajectories.

In any case $\lim e^{-\lambda_1 t}X = C_1 > 0$, $\lim Y = \bar{Y}$, $\lim Z = \bar{Z}$, $\lim W = \bar{W}$, then (4.15) holds again.■

Remark 10.1 *Finally there is no admissible trajectory converging to* $0 = (0,0,0,0)$, *or* $K_0 = (0,0,N+a,0)$, *or* $L_0 = (0,0,0,N+b)$. *Indeed the linearization at* 0 *gives*

$$X_t = -\frac{N-p}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a)Z, \quad W_t = (N+b)W$$

Then $\mathcal{V}_s$ *and* $\mathcal{V}_u$ *have dimension* 2, *hence* $\mathcal{V}_s$ *is contained in* $\{Z=W=0\}$, *and* $\mathcal{V}_u$ *in* $\{X=Y=0\}$. *The linearization at* K_0 *gives, with* $Z = N+a+\tilde{Z}$,

$$X_t = \frac{p+a}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a)\left[-sX - \delta Y - \tilde{Z}\right], \quad W_t = (N+b)W.$$

The eigenvalues are $\frac{p+a}{p-1}, -\frac{N-q}{q-1}, -(N+a), N+b$. *Then* $\mathcal{V}_s$ *and* $\mathcal{V}_u$ *have dimension* 2, *hence* $\mathcal{V}_s$ *is contained in* $\{Z=W=0\}$, *and* $\mathcal{V}_u$ *in* $\{Y=0\}$. *The case of* L_0 *follows by symmetry.*

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